Tikhonov Regularization for Inverse Medium Scattering in Banach Spaces

Frederic Weidling
joint work with Thorsten Hohage

Institute for Numerical and Applied Mathematics
Georg-August University of Göttingen

Computational Inverse Problems for Partial Differential Equations
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Outline

Schrödinger equation

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Given a potential q and an energy E find the solution to

$$(-\Delta + q)u = Eu \qquad \text{in } \mathbb{R}^3,$$

$$\lim_{|x|\to\infty} |x| \left(\frac{\partial}{\partial |x|} - i\sqrt{E} \right) u(x) = 0.$$

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Assumption on the potential q

- $q \in L^\infty$
- $\operatorname{supp} q \subset B(r)$
- q is absorbing, i.e. $\Im(q) \geq 0$

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} ensures unique solvability with $u \in H^1_{\mathrm{loc}}$

Near field inverse problem

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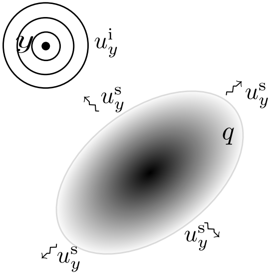
Incident fields are free space solutions

$$u_y^{\mathrm{i}}(x) = \frac{1}{4\pi} \frac{\mathrm{e}^{i\sqrt{E}|x-y|}}{|x-y|}.$$



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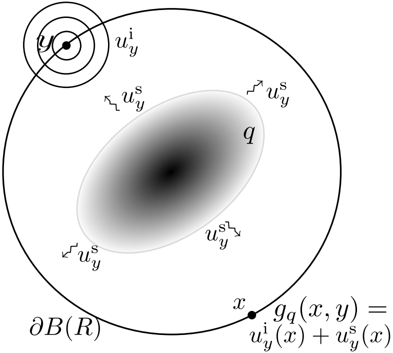
on $\partial B(R)$ with $R = |y| > r$.



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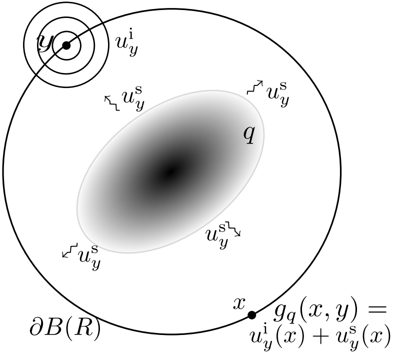
$\rightsquigarrow$ Repeat for all $y \in \partial B(R)$.



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Define operator $F\colon \operatorname{dom}(F) \to L^2(\partial B(R) \times \partial B(R)) =: \mathcal{Y}, q \mapsto g$

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$$q_\alpha^\delta \in \operatorname*{arg\,min}_{q \in \operatorname{dom}(F)} T_{\alpha,g^\delta}(q), \qquad T_{\alpha,g^\delta}(q) := \left[\frac{1}{2\alpha}\left\|F(q) - g^\delta \,\middle|\, \mathcal{Y}\right\|^2 + \mathcal{R}(q)\right]$$

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A variational source condition

$$\forall q : \quad \langle q^*, q^\dagger - q\rangle \leq \frac{1}{2}\Delta_{\mathcal{R}}(q, q^\dagger) + \psi\left(\left\|F(q) - F(q^\dagger) \,\middle|\, \mathcal{Y}\right\|^2\right)$$

with $q^* \in \partial\mathcal{R}(q^\dagger)$ implies rates of the form

$$\Delta_{\mathcal{R}}(q_\alpha^\delta, q^\dagger) \leq 4\psi(\delta^2)$$

for optimal choice of α.

Known results

- Exponential instability $\rightsquigarrow$ ψ must be of logarithmic form

N. Mandache. *Exponential instability in an inverse problem for the Schrödinger equation.* **Inverse Problems**, 17:1435–1444, 2001.

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$$\left\|q_1 - q_2 \mid L^2\right\| \leq AE^{1/2}\delta^{1/2} + B\left(E + \ln^2(\delta^{-1})\right)^{-s/3}$$

G. Alessandrini. *Stable determination of conductivity by boundary measurements.* **Applicable Analysis**, 27:153–172, 1988.

M. I. Isaev and R. G. Novikov. *Effectivized Hölder-logarithmic stability estimates for the Gel'fand inverse problem.* **Inverse Problems**, 30:095006, 2014.

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- Regularization strategy for $\mathcal{R}(q) = \|q \mid L^p\|^p$+constraint if $p > 3/2$
- VSC for $\mathcal{R}(q) = \|q \mid H^m\|^2$+constraint if $m > 3/2$ of the form

$$\psi(t) = A\left(\ln^2(\delta^{-1})\right)^{-\max\left\{1, \frac{s-m}{m+3/2}\right\}}$$

T. Hohage and F. Weidling. *Verification of a variational source condition for acoustic inverse medium scattering problems.* **Inverse Problems**, 31:075006, 2015.

A wishlist

- Penalty term $\mathcal{R}$
  - of the form $\mathcal{R}(q) = \frac{1}{r}\|q \,|\, \mathcal{X}\|^r$+constraints for some Banach space $\mathcal{X}$
  - not force solution smoothness
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- Regularization strategy
- Hölder-logarithmic w.r.t. energy E
- Unbounded exponent

How to verify a VSC?

Then $f^{\dagger}$ fulfills a VSC with index function

T. Hohage and F. Weidling. *Characterizations of Variational Source Conditions, Converse Results, and Maxisets of Spectral Regularization Methods.* **SIAM JNA**, 55:598–620, 2017.

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  - smoothness of the solution

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  - ill-posedness of the problem

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Norms

Norms on Besov space $B^s_{p,q}$ with $p \in (1,\infty), q \in [1,\infty], s \in \mathbb{R}$

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Fourier approach

$$f_j(x) := \mathcal{F}^*(\chi_j \mathcal{F} f)(x)$$

with $\chi_0(x)$ the characteristic function of the unit ball in $\mathbb{R}^3$

$$\chi_j(x) := \chi_0(2^{-j}x) - \chi_0(2^{-j+1}x)$$

for $j \in \mathbb{N}$

$$\left\| f \mid B^s_{p,q} \right\| := \left\{ \left[\sum_{j \in \mathbb{N}_0} 2^{jsq} \left\| f_j \mid L^p \right\|^q \right]^{\frac{1}{q}} \right.$$

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Wavelet approach

$$f(x) = \sum_{(j,m,l)\in I} \underbrace{\langle f, \phi^l_{j,m}\rangle}_{\lambda^l_{j,m}} \phi^l_{j,m}(x)$$

with $(\phi^l_{j,m})_{(j,m,l)\in I}$ a smooth normalized Daubechies wavelet system.

$$\left\| f \mid B^s_{p,q} \right\| := \begin{cases} \left[\sum_{j\in\mathbb{N}_0} 2^{jsq} \| f_j \mid L^p \|^q \right]^{\frac{1}{q}} \\ \left[\sum_{j\in\mathbb{N}_0} \sum_{l=1}^{L_j} 2^{jsq} 2^{jd(\frac{1}{2}-\frac{1}{p})q} \left(\sum_{m\in\mathbb{Z}^3} |\lambda^l_{j,m}|^p \right)^{\frac{q}{p}} \right]^{\frac{1}{q}} \end{cases}$$

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Properties

- Inclusions: for $s \in \mathbb{R}$, $\varepsilon > 0$ and $1 \leq r \leq q \leq \infty$

$$B^{s+\varepsilon}_{p,q} \subset B^{s+\varepsilon}_{p,\infty} \subset B^s_{p,1} \subset B^s_{p,r} \subset B^s_{p,q}$$

- Dual space:

$$\left(B^s_{p,q}\right)^* = B^{-s}_{p',q'}$$

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- Relation to L^p spaces:

$$B^0_{p,\min\{p,2\}} \subset L^p \subset B^0_{p,\max\{p,2\}}$$

- Relation to Sobolev spaces:

$$B^s_{p,p} = W^{s,p}, \qquad s \notin \mathbb{Z}$$

- Typical solutions to inverse problems: f smooth up to jumps $\rightsquigarrow f \in B^{d/p}_{p,\infty}$

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- Convexity: with Wavelet-norm

$$B^s_{p,q} \text{ is } \max\{2,p,q\}\text{-convex}$$

K. S. Kazimierski *On the smoothness and convexity of Besov spaces.* **Journal of Inverse and Ill-Posed Problems**, 21:411–429, 2013.

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Theorem

Let $f^* \in \partial \frac{1}{2} \| f^\dagger \mid B^0_{p,p} \|^2$ *then*

$$f^* = \sum_{(j,m,l) \in I} \mu^l_{j,m} \phi^l_{j,m}(x)$$

$$\text{with} \quad \mu^l_{j,m} = \left\| f^\dagger \mid B^0_{p,p} \right\|^{2-p} 2^{jd(\frac{p}{2}-1)} \frac{\lambda^l_{j,m}}{\left| \lambda^l_{j,m} \right|^{2-p}}.$$

In addition $f^\dagger \in B^s_{p,\infty}$ *for* $s > 0$ *if and only if* $f^* \in B^{s(p-1)}_{p',\infty}$.

Tikhonov functional

$$T_{\alpha,g^\delta}(q) = \frac{1}{2\alpha}\left\| F(q) - g^\delta \mid \mathcal{Y}\right\|^2 + \frac{1}{2}\left\| q \mid B^0_{p,p}\right\|^2 + \iota_{\{\Im(\cdot)\geq 0, \mathrm{supp}(\cdot)\subset B(r)\}}(q) + \iota_{\{\|\cdot \mid L^\infty\| \leq C_\infty\}}(q)$$

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This strategy is regularizing.

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Theorem

This strategy is regularizing.

Proof.

- Show: weak $B^0_{p,p}$-topology and weak L^2- topology conincide on $\{q \in B^0_{p,p} : \Im(q) \geq 0, \mathrm{supp}(q) \subset B(r), \|q \mid L^\infty\| \leq C_\infty\}$.
- Use results of:

A. Lechleiter, K. S. Kazimierski and M. Karamehmedović *Tikhonov regularization in L^p applied to inverse medium scattering.* **Inverse Problems**, 29:075003, 2013.

□

Main result

Theorem (Variational Source Condition)

$R > r > 0$, $E \geq 1$, $C_\infty > 0$, $2 \geq p > 1$, $s > 0$ and $C_s > 0$.
Let the true potential $q^\dagger$ satisfy:

$$\mathrm{supp}(q^\dagger) \subset B(r), \quad \Im(q^\dagger) \geq 0, \quad \|q^\dagger \mid L^\infty\| \leq C_\infty, \quad \|q^\dagger \mid B^s_{p,\infty}\| \leq C_s$$

Then $\exists c > 0$ such that for all $q \in \mathrm{dom}(T_{\alpha,\cdot})$ the VSC

$$\langle q^*, q^\dagger - q\rangle \leq \frac{1}{2}\Delta_{\frac{1}{2}\|\cdot \mid B^0_{p,p}\|^2}(q, q^\dagger) + cC_s(1 + C_s)\left(E^3\delta^{\frac{1}{2}} + (1 + C_\infty^2)\left(E + \ln^2(3 + \delta^{-2})\right)^{-\mu}\right)$$

holds true, where

$$\delta := \|F(q) - F(q^\dagger) \mid L^2\|, \qquad \mu = \min\left\{\frac{2}{4-p}, s(p-1)\right\}.$$

Corollaries

Corollary (Convergence rate)

Let q_α^δ be the solution of the Tikhonov functional for optimal α, then

$$\left\| q^\dagger - q_\alpha^\delta \mid B^0_{p,p} \right\| \lessapprox (1+C_s)\left(E^3 \delta^{\frac{1}{2}} + \left(1 + C_\infty^2\right)\left(E + \ln^2(3 + \delta^{-2})\right)^{-\mu}\right)^{1/2}$$

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Corollary (Stability)

Let q_1, q_2 fulfill the conditions on $q^\dagger$, then

$$\left\|q_1 - q_2 \mid B^0_{p,p}\right\| \lessapprox (1+C_s)\left(E^3\delta^{\frac{1}{2}} + \left(1 + C_\infty^2\right)\left(E + \ln^2(3+\delta^{-2})\right)^{-\mu}\right)^{1/2}$$

Projection choice

By our strategy we need:

- A choice of projection that makes use of Besov smoothness

$$\|(I - P_k)f^* \,|\, \mathcal{X}^*\| \leq \kappa(k), \qquad \inf_{k \in K} \kappa(k) = 0$$

$\rightsquigarrow$ Fourier based projection or
$\rightsquigarrow$ Wavelet based projection

- Characterization of ill-posedness

$$\langle f^*, P_k^*(f^\dagger - f)\rangle \leq \sigma(k)\|F(f^\dagger) - F(f) \,|\, \mathcal{Y}\| + \gamma\kappa(k)\|f^\dagger - f \,|\, \mathcal{X}\|$$

$\rightsquigarrow$ Fourier based projection

Smoothness result

Define the projections:

$$\tilde{P}_k f := \mathcal{F}^* \chi_{\{|\cdot| \leq 2^k\}} \mathcal{F} f, \qquad k \in \mathbb{N}$$

$$P_\Omega \phi^l_{j,m} := \begin{cases} \phi^l_{j,m} & \text{supp}(\phi^l_{j,m}) \cap B(r) \neq \emptyset \\ 0 & \text{supp}(\phi^l_{j,m}) \cap B(r) = \emptyset \end{cases}$$

Set $P_k = P_\Omega \tilde{P}_k P_\Omega$

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One obtains:

$$\left\| (I - P_k) q^* \mid B^0_{p',p'} \right\| \leq \underbrace{c \left(2^k\right)^{-s(p-1)} \left\| q^* \mid B^{s(p-1)}_{p',\infty} \right\|}_{=:\kappa(k)}$$

Brief version

Use smoothness of q^* and properties of P_Ω to show:

$$\langle q^*, P_k^*(q^\dagger - q)\rangle \leq cC_s \left\| \chi_{\{|\cdot|\leq 2^k\}} \mathcal{F}(q^\dagger - q) \,\middle|\, L^\infty \right\|$$

Brief version

Use smoothness of q^* and properties of P_Ω to show:

$$\langle q^*, P_k^*(q^\dagger - q)\rangle \leq cC_s \left\| \chi_{\{|\cdot|\leq 2^k\}} \mathcal{F}(q^\dagger - q) \,\middle|\, L^\infty \right\|$$

Treatable with "standard machinary" of scattering theory:

Lemma

$$\left\| \chi_{\{|\cdot|\leq 2^k\}} \mathcal{F}(q^\dagger - q) \,\middle|\, L^\infty \right\| \leq c\left(E^3 \mathrm{e}^{(2R+1)t}\delta + \frac{C_\infty}{\sqrt{E+2t^2}} \left\| q^\dagger - q \,\middle|\, L^2 \right\| \right).$$

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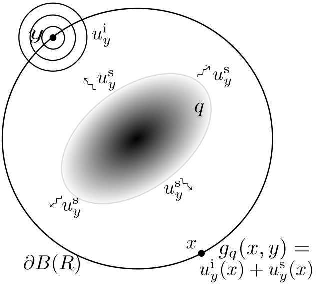
Summary

Method to recover q from g



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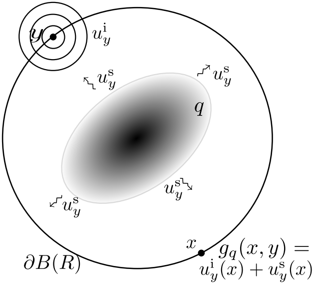

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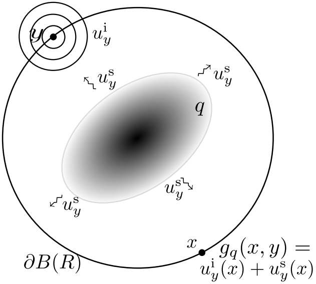

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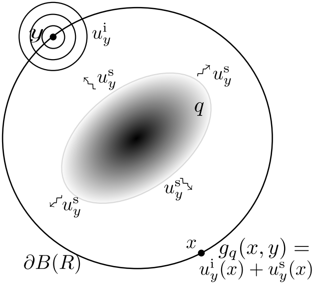

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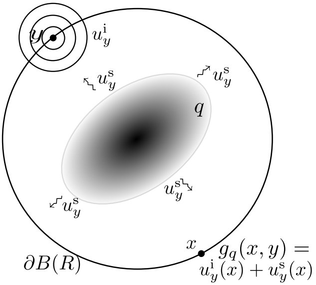

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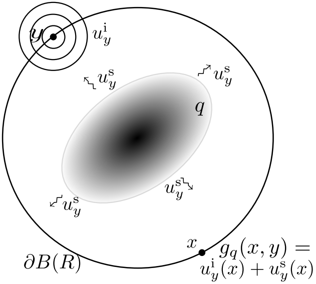

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Thank you for your attention!