Characterizations of Variational Source Conditions, Converse Results and Maxisets of Spectral Regularization Methods

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joint work with Thorsten Hohage

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Outline

Classical Setup

Setup:

- Let $\mathbb{X}, \mathbb{Y}$ be Hilbert spaces
- $T : \mathbb{X} \to \mathbb{Y}$ be a linear operator
- $f^\dagger \in \mathbb{X}$ the true solution
- noisy measurement g^{obs}

$$g^{\mathrm{obs}} = Tf^\dagger + \xi, \qquad \|\xi\| \leq \delta$$

Problem: find approximation of $f^\dagger$ from g^{obs}, but $T^\dagger$ unbounded

Assumptions on spectral regularization

$$f_\alpha^\delta := R_\alpha g^{\mathrm{obs}} \qquad \text{with} \qquad R_\alpha = q_\alpha(T^*T)T^*$$

Assumptions on SR

With $r_\alpha(\lambda) := 1 - \lambda q_\alpha(\lambda)$ assume that for all $\lambda \in \sigma(T^*T)$ and $0 < \alpha \le \overline{\alpha}$

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1. $|q_\alpha(\lambda)| \leq \frac{C_1}{\alpha}$ for some $C_1 > 0$,
2. $\lambda \mapsto r_\alpha(\lambda)$ is decreasing and $r_\alpha(\lambda) \geq 0$,
3. $\lim_{\alpha \to 0} r_\alpha(\lambda) = 0$ and

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4. $\alpha \mapsto r_\alpha(\lambda)$ is increasing,
5. $0 < C_2 \leq \sup_{0<\alpha\leq\overline{\alpha}} r_\alpha(\alpha) \leq C_3 < 1$.

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include	exclude
k-iterated Tikhonov	spectral cut-off
Landweber	ν-methods
Showalter	
Lardy	
modified spectral cut-off	

Rate of convergence

$$\left\| f^\dagger - f_\alpha^\delta \right\|^2 \leq \psi(\delta^2)$$

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- Spectral source conditions for an index function κ:

$$f^\dagger = \kappa\left(T^*T\right)\omega, \qquad \|\omega\| \leq \rho$$

$$\rightsquigarrow \quad \psi_\kappa(\delta^2) = 4\rho^2\kappa\left(\Theta^{-1}\left(\frac{\delta}{\rho}\right)\right)^2, \qquad \Theta(t) := \sqrt{t}\kappa(t).$$

- Variational source conditions (VSC) for a concave index function ψ:

$$\forall f: \quad 4\left\langle f^\dagger, f^\dagger - f\right\rangle_{\mathbb{X}} \leq \left\|f^\dagger - f\right\|_{\mathbb{X}}^2 + \psi\left(\left\|F(f) - F(f^\dagger)\right\|_{\mathbb{Y}}^2\right)$$

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How to verify such a condition?

General strategy for verification

Let $P_r \in \mathcal{L}(\mathbb{X})$ be a family of orthogonal projection operators such that for all r

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$$\left\langle f^\dagger, P_r(f^\dagger - f)\right\rangle \leq \sigma(r)\left\|Tf - Tf^\dagger\right\|_{\mathbb{Y}} + C\kappa(r)\left\|f - f^\dagger\right\|_{\mathbb{X}},$$

for all f with $\|f - f^\dagger\| \leq 4\|f^\dagger\|$.

Compare to Lipschitz stability estimates for T^{-1}:

$$\left\|P_r(f^\dagger - f)\right\|_{\mathbb{X}} \leq \tilde{\sigma}(r)\left\|TP_rf^\dagger - TP_rf\right\|_{\mathbb{Y}}$$

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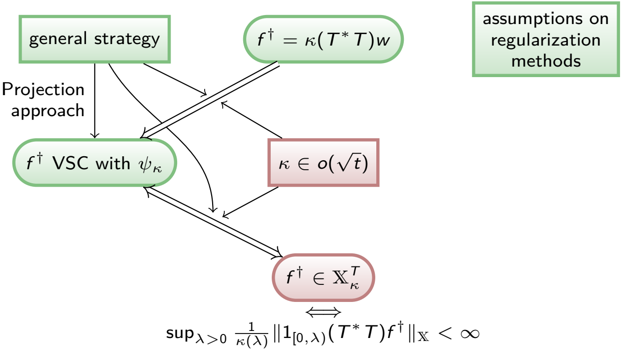
Then $f^\dagger$ fulfills a variational source condition with

$$\psi(t) := 4 \inf_r \left[(C+1)^2\kappa(r)^2 + \sigma(r)\sqrt{t}\right].$$

Summary and plan



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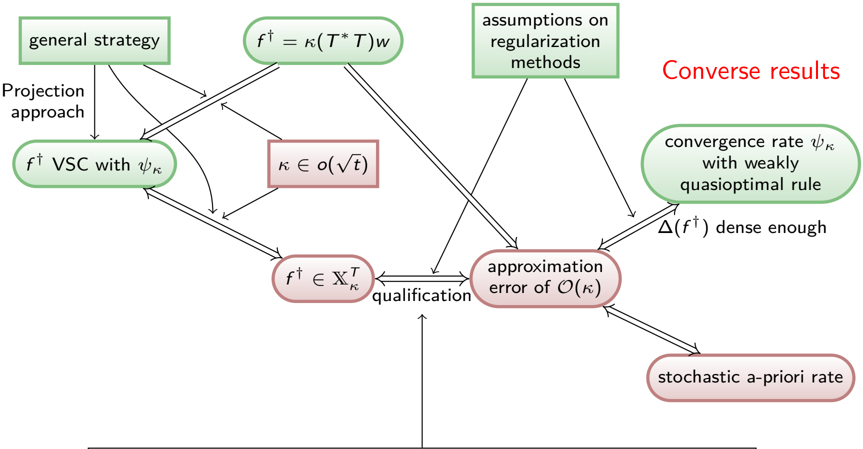

V. Albani, P. Elbau, M.V. de Hoop, O. Scherzer.
Optimal convergence rates results for linear inverse problems in Hilbert spaces.
Numerical Functional Analysis and Optimization 37:521–540, 2016.

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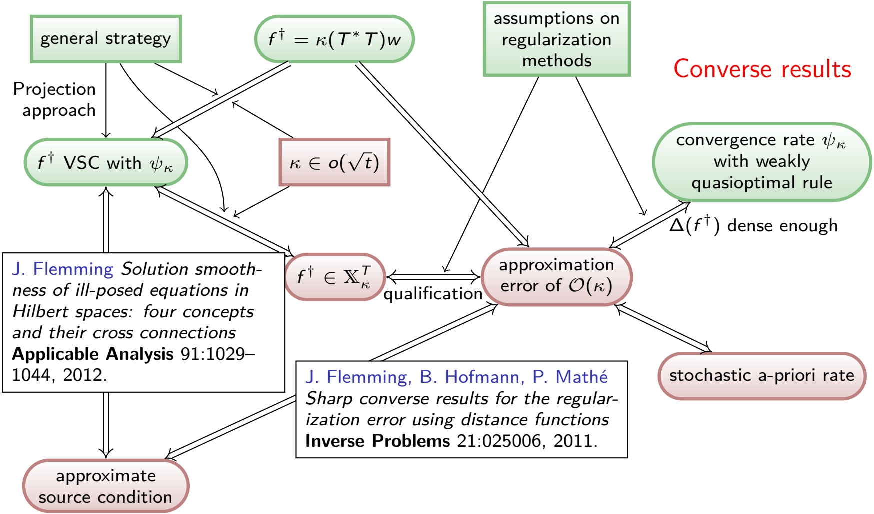

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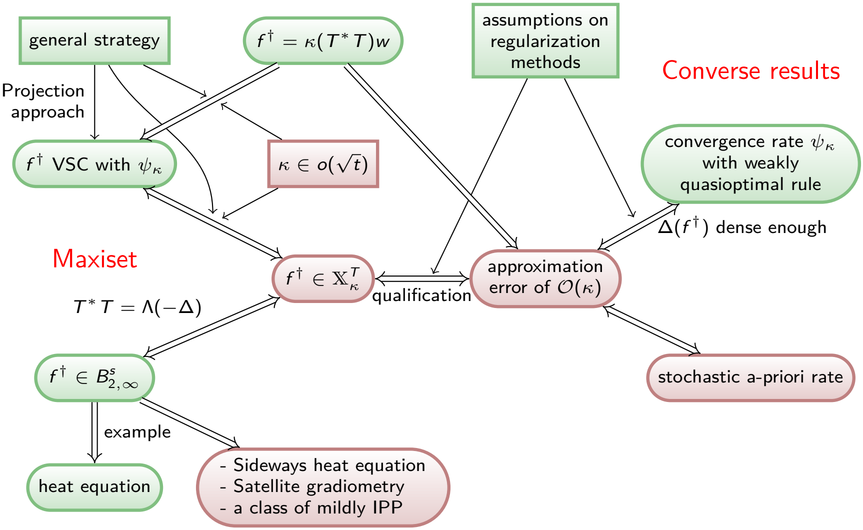

Parameter choice rules

A parameter choice rule α_* is called

- weakly quasioptimal

- strongly quasioptimal

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Examples:

- discrepancy principle: strongly quasioptimal for methods with infinite qualification
- Lepskiĭ: weakly quasioptimal

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Interchangeability result

Lemma

For all $\delta \in \Delta(f^\dagger) := \{\|r_\alpha(T^*T)f^\dagger\|/\|R_\alpha\| : 0 < \alpha < \overline{\alpha}\}$ *we have*

$$\inf_{0<\alpha<\overline{\alpha}} \sup_{\|\xi\|\leq\delta} \left\|R_\alpha(Tf^\dagger + \xi) - f^\dagger\right\| \leq 2\sqrt{2} \sup_{\|\xi\|\leq\delta} \inf_{0<\alpha\leq\overline{\alpha}} \left\|R_\alpha(Tf^\dagger + \xi) - f^\dagger\right\|$$

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Corollary:

- In many cases weak and strong quasioptimality coincide.

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Theorem

Let $\kappa(r\alpha) \leq r^p\kappa(\alpha)$ *for some* $p \geq 1$ *and all* $r \geq 1$. *Then for any finite* $\delta_0 > 0$ *the following is equivalent for all considered methods, all* $f^\dagger \neq 0$, *and all weakly quasioptimal parameter choice rules* α_**:*

1. $\sup_{0<\alpha\leq\overline{\alpha}} \frac{1}{\kappa(\alpha)^2}\|r_\alpha(T^*T)f^\dagger\|^2 < \infty$.
2. $\sup_{0<\delta\leq\delta_0} \frac{1}{\psi_\kappa(\delta^2)} \sup_{\|\xi\|\leq\delta} \left\|R_{\alpha_*}(Tf^\dagger + \xi) - f^\dagger\right\|^2 < \infty$.

Besov spaces

Maxisets: largest set on which a given methods achieves a given rate of convergence $\rightsquigarrow \mathbb{X}_{\kappa}^{T}$ is maxiset

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Theorem

Let Δ be a Laplace-Beltrami operator on Ω ("sufficiently nice"), $\Lambda : [0, \infty) \to (0, \infty)$ continuous and monotonically decreasing with $\lim_{\mu \to \infty} \Lambda(\mu) = 0$. Let $T : \mathbb{X} := L^2(\Omega) \to \mathbb{Y}$ be bounded such that

$$T^*T = \Lambda(-\Delta) \quad \text{and set} \quad \kappa(\alpha) = (\Lambda^{-1}(\alpha))^{-1/2}$$

Then $\mathbb{X}_{\kappa^s}^{T} = B_{2,\infty}^{s}(\Omega)$ for all $s > 0$ with equivalent norms.

Proof based on:

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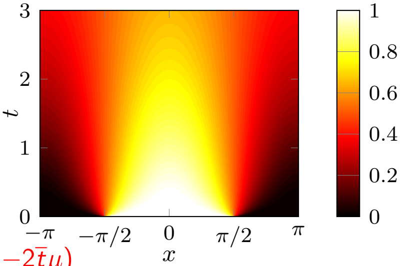
Besov spaces $B_{2,\infty}^s$

- K-interpolation spaces of Sobolev spaces H^n
- embedding property: $H^s \subset B_{2,\infty}^s \subset H^{s-\varepsilon}$ for all $0 < \varepsilon < s$.

Backward heat equation

$$\partial_t u = \Delta u \qquad \text{in } \mathbb{S}^1 \times (0, \bar{t})$$
$$u(\cdot, 0) = f \qquad \text{on } \mathbb{S}^1$$

unknown: initial temperature f
observations: $g = u(\cdot, \bar{t})$, final temperature



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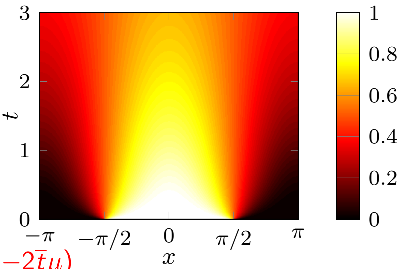

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Theorem

The following statements are equivalent for $\beta > 0$ and $f^\dagger \neq 0$:

1. $f^\dagger \in B^{2\beta}_{2,\infty}(\mathbb{S}^1)$
2. *$f^\dagger$ satisfies a VSC with $\psi(t) = C \log(3 + t^{-1})^{-2\beta}$, $C > 0$.*
3. *For a weakly quasioptimal parameter choice rule α_* we have* $\sup\{\left\|R_{\alpha_*} g^{\text{obs}} - f^\dagger\right\| : \left\|g^{\text{obs}} - Tf^\dagger\right\| \leq \delta\} = \mathcal{O}\left(\log(\delta^{-1})^{-\beta}\right)$

Relation to spectral source conditions

Characterization of spectral source conditions known:

$$f^\dagger = \varphi_\beta(T^*T)w, \quad \varphi_\beta(\lambda) = (-\ln\lambda)^{-\beta} \quad \Leftrightarrow \quad f^\dagger \in H^{2\beta}(\mathbb{S}^1)$$

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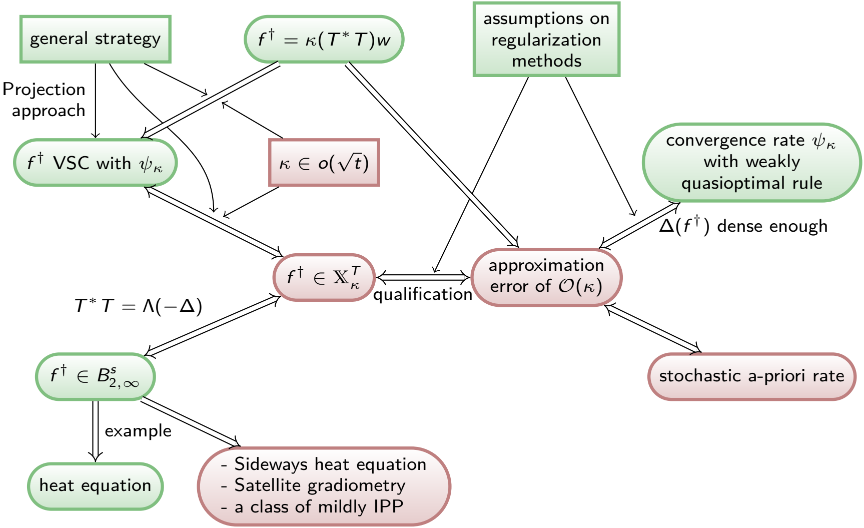
For given example:

$$f^\dagger(t) = \begin{cases} 1, & |t| < \frac{\pi}{2}, \\ 0, & \text{else} \end{cases} \quad \Longrightarrow \quad \widehat{f^\dagger}(n) \approx \frac{1}{|n|}$$

$$\Longrightarrow \quad f^\dagger \in \begin{cases} H^{2\beta}, & \text{for } \beta \in [0, 1/4) \Longrightarrow \text{rate of } \mathcal{O}\left(\log(\delta^{-1})^{-\beta}\right) \\ B^{1/2}_{2,\infty}, & \Longrightarrow \text{rate of } \mathcal{O}\left(\log(\delta^{-1})^{-1/4}\right) \end{cases}$$

Summary


general strategy
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assumptions on regularization methods
Projection approach
$f^\dagger$ VSC with ψ_κ
$\kappa \in o(\sqrt{t})$
convergence rate ψ_κ with weakly quasioptimal rule
$\Delta(f^\dagger)$ dense enough
$f^\dagger \in \mathbb{X}_\kappa^T$
qualification
approximation error of $\mathcal{O}(\kappa)$
$T^*T = \Lambda(-\Delta)$
$f^\dagger \in B^s_{2,\infty}$
stochastic a-priori rate
example
heat equation
- Sideways heat equation
- Satellite gradiometry
- a class of mildly IPP

Summary

Thank you for your attention!

available at arXiv:1603.05133