Tikhonov Regularization for Inverse Medium Scattering in Banach Spaces

Frederic Weidling[1]
joint work with Thorsten Hohage

Institute for Numerical and Applied Mathematics
Georg-August University of Göttingen

IMA Conference on Inverse Problems from Theory to Application
September 20, 2017

[1]finanical support by CRC 755

Outline

Schrödinger equation

Schrödinger equation

Given a potential q and an energy E find the solution to

$$(-\Delta + q)u = Eu \qquad \text{in } \mathbb{R}^3,$$

$$\lim_{|x|\to\infty} |x| \left(\frac{\partial}{\partial|x|} - i\sqrt{E} \right) u(x) = 0.$$

Schrödinger equation

Schrödinger equation

Given a potential q and an energy E find the solution to

$$(-\Delta + q)u = Eu \qquad \text{in } \mathbb{R}^3,$$
$$\lim_{|x|\to\infty} |x| \left(\frac{\partial}{\partial |x|} - i\sqrt{E} \right) u(x) = 0.$$

Assumption on the potential q

- $q \in L^\infty$
- $\operatorname{supp} q \subset B(r)$
- q is absorbing, i.e. $\Im(q) \geq 0$

Schrödinger equation

Schrödinger equation

Given a potential q and an energy E find the solution to

$$\begin{aligned} (-\Delta + q)u &= Eu && \text{in } \mathbb{R}^3, \\ \lim_{|x|\to\infty} |x| \left(\frac{\partial}{\partial |x|} - i\sqrt{E} \right) u(x) &= 0. \end{aligned}$$

Assumption on the potential q

- $q \in L^\infty$
- $\operatorname{supp} q \subset B(r)$
- q is absorbing, i.e. $\Im(q) \geq 0$

} ensures unique solvability with $u \in H^1_{\mathrm{loc}}$

Near field inverse problem

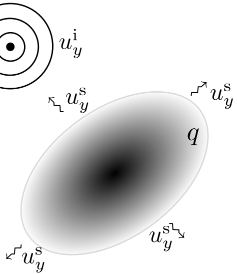
$$\text{Total field } u = \underbrace{\text{incident field } u_y^{\mathrm{i}}}_{\text{known}} + \underbrace{\text{scattered field } u_y^{\mathrm{s}}}_{\text{unknown}}$$

Near field inverse problem

$$\text{Total field } u = \underbrace{\text{incident field } u_y^{\mathrm{i}}}_{\text{known}} + \underbrace{\text{scattered field } u_y^{\mathrm{s}}}_{\text{unknown}}$$

Incident fields are free space solutions

$$u_y^{\mathrm{i}}(x) = \frac{1}{4\pi}\frac{\mathrm{e}^{i\sqrt{E}|x-y|}}{|x-y|}.$$



Near field inverse problem

$$\text{Total field } u = \underbrace{\text{incident field } u^{\mathrm{i}}_y}_{\text{known}} + \underbrace{\text{scattered field } u^{\mathrm{s}}_y}_{\text{unknown}}$$

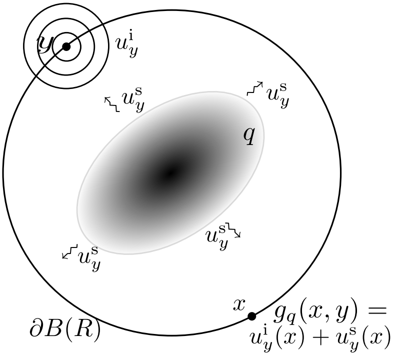
Incident fields are free space solutions

$$u^{\mathrm{i}}_y(x) = \frac{1}{4\pi} \frac{\mathrm{e}^{i\sqrt{E}|x-y|}}{|x-y|}.$$

Measurement is the Green's function

$$g_q(x,y) = u^{\mathrm{i}}_y(x) + u^{\mathrm{s}}_y(x)$$

on $\partial B(R)$ with $R = |y| > r$.



Near field inverse problem

$$\text{Total field } u = \underbrace{\text{incident field } u_y^{\mathrm{i}}}_{\text{known}} + \underbrace{\text{scattered field } u_y^{\mathrm{s}}}_{\text{unknown}}$$

Incident fields are free space solutions

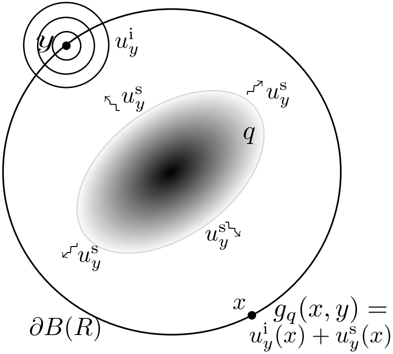
$$u_y^{\mathrm{i}}(x) = \frac{1}{4\pi}\frac{\mathrm{e}^{i\sqrt{E}|x-y|}}{|x-y|}.$$

Measurement is the Green's function

$$g_q(x,y) = u_y^{\mathrm{i}}(x) + u_y^{\mathrm{s}}(x)$$

on $\partial B(R)$ with $R = |y| > r$.

$\rightsquigarrow$ Repeat for all $y \in \partial B(R)$.



Solution strategy

Define operator $F\colon \operatorname{dom}(F) \to L^2(\partial B(R) \times \partial B(R)) =: \mathcal{Y}, q \mapsto g$

Solution strategy

Define operator $F\colon \operatorname{dom}(F) \to L^2(\partial B(R) \times \partial B(R)) =: \mathcal{Y}, q \mapsto g$

Problem is ill-posed $\rightsquigarrow$ apply Tikhonov regularization of the form

$$q_\alpha^\delta \in \operatorname*{arg\,min}_{q \in \operatorname{dom}(F)} T_{\alpha, g^\delta}(q), \qquad T_{\alpha, g^\delta}(q) := \left[\frac{1}{2\alpha} \left\| F(q) - g^\delta \,\middle|\, \mathcal{Y} \right\|^2 + \mathcal{R}(q) \right]$$

Solution strategy

Define operator $F\colon \operatorname{dom}(F) \to L^2(\partial B(R) \times \partial B(R)) =: \mathcal{Y}, q \mapsto g$

Problem is ill-posed $\rightsquigarrow$ apply Tikhonov regularization of the form

$$q_\alpha^\delta \in \operatorname*{arg\,min}_{q \in \operatorname{dom}(F)} T_{\alpha, g^\delta}(q), \qquad T_{\alpha, g^\delta}(q) := \left[\frac{1}{2\alpha} \left\| F(q) - g^\delta \,\middle|\, \mathcal{Y} \right\|^2 + \mathcal{R}(q) \right]$$

Questions:

- Is this a regularization strategy?
  $\rightsquigarrow$ depends on $\mathcal{R}$

Solution strategy

Define operator $F\colon \operatorname{dom}(F) \to L^2(\partial B(R) \times \partial B(R)) =: \mathcal{Y}, q \mapsto g$

Problem is ill-posed $\rightsquigarrow$ apply Tikhonov regularization of the form

$$q_\alpha^\delta \in \operatorname*{arg\,min}_{q \in \operatorname{dom}(F)} T_{\alpha,g^\delta}(q), \qquad T_{\alpha,g^\delta}(q) := \left[\frac{1}{2\alpha}\left\|F(q) - g^\delta \,\middle|\, \mathcal{Y}\right\|^2 + \mathcal{R}(q)\right]$$

Questions:

- Is this a regularization strategy?
  $\rightsquigarrow$ depends on $\mathcal{R}$
- How good does q_α^δ approximate $q^\dagger$?
  $\rightsquigarrow$ Use a variational source condition

$$\forall q : \quad \langle q^*, q^\dagger - q\rangle \leq \frac{1}{2}\Delta_{\mathcal{R}}(q, q^\dagger) + \psi\left(\left\|F(q) - F(q^\dagger) \,\middle|\, \mathcal{Y}\right\|^2\right)$$

with $q^* \in \partial\mathcal{R}(q^\dagger)$ to obtain a rate of the form

$$\Delta_{\mathcal{R}}(q_\alpha^\delta, q^\dagger) \leq 4\psi(\delta^2)$$

for optimal choice of α.

Known results

- Exponential instability $\rightsquigarrow$ ψ must be of logarithmic form

N. Mandache. *Exponential instability in an inverse problem for the Schrödinger equation.* **Inverse Problems**, 17:1435–1444, 2001.

Known results

- Exponential instability $\rightsquigarrow$ ψ must be of logarithmic form
- Conditional Stability: $\|q_j \mid L^\infty\| \leq c_1$ and $\|q_j \mid H^s\| \leq c_2$:

$$\left\|q_1 - q_2 \mid L^2\right\| \leq AE^{1/2}\delta^{1/2} + B\left(E + \ln^2(\delta^{-1})\right)^{-s/3}$$

G. Alessandrini. *Stable determination of conductivity by boundary measurements.* **Applicable Analysis**, 27:153–172, 1988.

M. I. Isaev and R. G. Novikov. *Effectivized Hölder-logarithmic stability estimates for the Gel'fand inverse problem.* **Inverse Problems**, 30:095006, 2014.

Known results

- Exponential instability $\rightsquigarrow$ ψ must be of logarithmic form
- Conditional Stability: $\|q_j \mid L^\infty\| \leq c_1$ and $\|q_j \mid H^s\| \leq c_2$:

$$\left\|q_1 - q_2 \mid L^2\right\| \leq AE^{1/2}\delta^{1/2} + B\big(E + \ln^2(\delta^{-1})\big)^{-s/3}$$

- Regularization strategy for $\mathcal{R}(q) = \|q \mid L^p\|^p$+constraint if $p > 3/2$

A. Lechleiter, K. S. Kazimierski and M. Karamehmedović *Tikhonov regularization in L^p applied to inverse medium scattering.* **Inverse Problems**, 29:075003, 2013.

Known results

- Exponential instability $\rightsquigarrow \psi$ must be of logarithmic form
- Conditional Stability: $\|q_j \mid L^\infty\| \le c_1$ and $\|q_j \mid H^s\| \le c_2$:

$$\left\|q_1 - q_2 \mid L^2\right\| \le AE^{1/2}\delta^{1/2} + B\left(E + \ln^2(\delta^{-1})\right)^{-s/3}$$

- Regularization strategy for $\mathcal{R}(q) = \|q \mid L^p\|^p$+constraint if $p > 3/2$
- VSC for $\mathcal{R}(q) = \|q \mid H^m\|^2$+constraint if $m > 3/2$ of the form

$$\psi(t) = A\left(\ln^2(\delta^{-1})\right)^{-\max\left\{1, \frac{s-m}{m+3/2}\right\}}$$

frist verification of a vsc for nonlinear pde but enforces smoothness

T. Hohage and F. Weidling. *Verification of a variational source condition for acoustic inverse medium scattering problems.* **Inverse Problems**, 31:075006, 2015.

A wishlist

- Penalty term $\mathcal{R}$
  - of the form $\mathcal{R}(q) = \frac{1}{r}\|q \mid \mathcal{X}\|^r$+constraints for some Banach space $\mathcal{X}$
  - not force solution smoothness
  - sparsity enforcing (i.e. p close to 1)

A wishlist

- Penalty term $\mathcal{R}$
  - of the form $\mathcal{R}(q) = \frac{1}{r}\|q \mid \mathcal{X}\|^r$+constraints for some Banach space $\mathcal{X}$
  - not force solution smoothness
  - sparsity enforcing (i.e. p close to 1)
- Regularization strategy

A wishlist

- Penalty term $\mathcal{R}$
  - of the form $\mathcal{R}(q) = \frac{1}{r}\|q \mid \mathcal{X}\|^r$+constraints for some Banach space $\mathcal{X}$
  - not force solution smoothness
  - sparsity enforcing (i.e. p close to 1)
- Regularization strategy
- variational source condition with ψ
  - Hölder-logarithmic w.r.t. energy E
  - Unbounded exponent

How to verify a VSC?

Then $f^\dagger$ fulfills a VSC with index function

T. Hohage and F. Weidling. *Characterizations of Variational Source Conditions, Converse Results, and Maxisets of Spectral Regularization Methods.* **SIAM JNA**, 55:598–620, 2017.

How to verify a VSC?

- Assume r-convexity of $\mathcal{X}$, then

$$\frac{C_\Delta}{r}\|f_1 - f_2 \,|\, \mathcal{X}\|^r \leq \Delta_{\frac{1}{r}\|\cdot\,|\,\mathcal{X}\|^r}(f_2, f_1)$$

Then $f^\dagger$ fulfills a VSC with index function

T. Hohage and F. Weidling. *Characterizations of Variational Source Conditions, Converse Results, and Maxisets of Spectral Regularization Methods.* **SIAM JNA**, 55:598–620, 2017.

How to verify a VSC?

- Assume r-convexity of $\mathcal{X}$, then

$$\frac{C_\Delta}{r}\|f_1 - f_2 \,|\, \mathcal{X}\|^r \leq \Delta_{\frac{1}{r}\|\cdot\,|\,\mathcal{X}\|^r}(f_2, f_1)$$

- Let $f^* \in \partial \frac{1}{r}\|f^\dagger \,|\, \mathcal{X}\|^r$, $P_k \colon \mathcal{X}^* \to \mathcal{X}^*$ and quantify
  - smoothness of the solution

$$\|(I - P_k)f^* \,|\, \mathcal{X}^*\| \leq \kappa(k), \qquad \inf_{k\in K} \kappa(k) = 0$$

  - ill-posedness of the problem

$$\left\langle f^*, P_k^*(f^\dagger - f)\right\rangle \leq \sigma(k)\left\|F(f^\dagger) - F(f)\,\middle|\,\mathcal{Y}\right\| + \gamma\kappa(k)\left\|f^\dagger - f\,\middle|\,\mathcal{X}\right\|$$

Then $f^\dagger$ fulfills a VSC with index function

T. Hohage and F. Weidling. *Characterizations of Variational Source Conditions, Converse Results, and Maxisets of Spectral Regularization Methods.* **SIAM JNA**, 55:598–620, 2017.

How to verify a VSC?

- Assume r-convexity of $\mathcal{X}$, then

$$\frac{C_\Delta}{r}\|f_1 - f_2 \,|\, \mathcal{X}\|^r \leq \Delta_{\frac{1}{r}\|\cdot\,|\,\mathcal{X}\|^r}(f_2, f_1)$$

- Let $f^* \in \partial \frac{1}{r}\|f^\dagger \,|\, \mathcal{X}\|^r$, $P_k\colon \mathcal{X}^* \to \mathcal{X}^*$ and quantify
  - smoothness of the solution

$$\|(I - P_k)f^* \,|\, \mathcal{X}^*\| \leq \kappa(k), \qquad \inf_{k\in K} \kappa(k) = 0$$

  - ill-posedness of the problem

$$\left\langle f^*, P_k^*(f^\dagger - f)\right\rangle \leq \sigma(k)\left\|F(f^\dagger) - F(f) \,\middle|\, \mathcal{Y}\right\| + \gamma\kappa(k)\left\|f^\dagger - f \,\middle|\, \mathcal{X}\right\|$$

Then $f^\dagger$ fulfills a VSC with index function

$$\psi_{\mathrm{vsc}}(t) = \inf_{k\in K}\left[\sigma(k)\sqrt{t} + \frac{1}{r'}\left(\frac{2}{C_\Delta}\right)^{\frac{r'}{r}}(1+\gamma)^{r'}\kappa(k)^{r'}\right].$$

T. Hohage and F. Weidling. *Characterizations of Variational Source Conditions, Converse Results, and Maxisets of Spectral Regularization Methods.* **SIAM JNA**, 55:598–620, 2017.

How to verify a VSC?

- Assume r-convexity of $\mathcal{X}$, then

$$\frac{C_\Delta}{r}\|f_1 - f_2 \,|\, \mathcal{X}\|^r \leq \Delta_{\frac{1}{r}\|\cdot\,|\,\mathcal{X}\|^r}(f_2, f_1)$$

- Let $f^* \in \partial \frac{1}{r}\|f^\dagger \,|\, \mathcal{X}\|^r$, $P_k \colon \mathcal{X}^* \to \mathcal{X}^*$ and quantify
  - smoothness of the solution

$$\|(I - P_k)f^* \,|\, \mathcal{X}^*\| \leq \kappa(k), \qquad \inf_{k\in K} \kappa(k) = 0$$

  - ill-posedness of the problem

$$\left\langle f^*, P_k^*(f^\dagger - f)\right\rangle \leq \sigma(k)\left\|F(f^\dagger) - F(f)\,\middle|\,\mathcal{Y}\right\| + \gamma\kappa(k)\left\|f^\dagger - f\,\middle|\,\mathcal{X}\right\|$$

Then $f^\dagger$ fulfills a VSC with index function

$$\psi_{\mathrm{vsc}}(t) = \inf_{k\in K}\left[\sigma(k)\sqrt{t} + \frac{1}{r'}\left(\frac{2}{C_\Delta}\right)^{\frac{r'}{r}}(1+\gamma)^{r'}\kappa(k)^{r'}\right].$$

T. Hohage and F. Weidling. *Characterizations of Variational Source Conditions, Converse Results, and Maxisets of Spectral Regularization Methods.* **SIAM JNA**, 55:598–620, 2017.

Example: Compact Operators

Let $\mathcal{X}, \mathcal{Y}$ be Hilbert spaces, F compact with singular system $(f_j, g_j, \sigma_j)_{j \in \mathbb{N}}$

$$P_k f = \sum_{j \leq k} \langle f, f_j \rangle f_j \qquad Q_k g = \sum_{j \leq k} \langle g, g_j \rangle g_j$$

Example: Compact Operators

Let $\mathcal{X}, \mathcal{Y}$ be Hilbert spaces, F compact with singular system $(f_j, g_j, \sigma_j)_{j \in \mathbb{N}}$

$$P_k f = \sum_{j \leq k} \langle f, f_j \rangle f_j \qquad Q_k g = \sum_{j \leq k} \langle g, g_j \rangle g_j$$

- Smoothness: How fast does

$$\|(I - P_k) f^\dagger \mid \mathcal{X}\|^2 = \sum_{j > k} |\langle f^\dagger, f_j \rangle|^2 = \kappa(k)$$

decay $\rightsquigarrow$ smoothness of $f^\dagger$

Example: Compact Operators

Let $\mathcal{X}, \mathcal{Y}$ be Hilbert spaces, F compact with singular system $(f_j, g_j, \sigma_j)_{j \in \mathbb{N}}$

$$P_k f = \sum_{j \leq k} \langle f, f_j \rangle f_j \qquad Q_k g = \sum_{j \leq k} \langle g, g_j \rangle g_j$$

- Smoothness: How fast does

$$\|(I - P_k) f^\dagger \,|\, \mathcal{X}\|^2 = \sum_{j > k} |\langle f^\dagger, f_j \rangle|^2 = \kappa(k)$$

  decay $\rightsquigarrow$ smoothness of $f^\dagger$

- Ill-posedness:

$$\begin{aligned} \langle f^\dagger, P_k(f^\dagger - f) \rangle &\leq \|f^\dagger \,|\, \mathcal{X}\| \|P_k(f^\dagger - f) \,|\, \mathcal{X}\| \\ &\leq \|f^\dagger \,|\, \mathcal{X}\| \frac{1}{\sigma_k} \|Q_k(F f^\dagger - Ff) \,|\, \mathcal{Y}\| \end{aligned}$$

Example: Compact Operators

Let $\mathcal{X}, \mathcal{Y}$ be Hilbert spaces, F compact with singular system $(f_j, g_j, \sigma_j)_{j\in\mathbb{N}}$

$$P_k f = \sum_{j\leq k} \langle f, f_j\rangle f_j \qquad Q_k g = \sum_{j\leq k} \langle g, g_j\rangle g_j$$

- Smoothness: How fast does

$$\|(I - P_k)f^\dagger \,|\, \mathcal{X}\|^2 = \sum_{j>k} |\langle f^\dagger, f_j\rangle|^2 = \kappa(k)$$

decay $\rightsquigarrow$ smoothness of $f^\dagger$

- Ill-posedness:

$$\begin{aligned}\langle f^\dagger, P_k(f^\dagger - f)\rangle &\leq \|f^\dagger \,|\, \mathcal{X}\| \|P_k(f^\dagger - f) \,|\, \mathcal{X}\| \\ &\leq \underbrace{\|f^\dagger \,|\, \mathcal{X}\| \frac{1}{\sigma_k}}_{=\sigma(k)} \|\cancel{Q_k}(Ff^\dagger - Ff) \,|\, \mathcal{Y}\|\end{aligned}$$

decay of singular values $\widehat{=}$ degree of ill-posedness

Besov spaces as candidates for $\mathcal{X}$

Besov space $B^s_{p,q}$

$$\begin{cases} p \in [1,\infty] & \text{integrability} \\ q \in [1,\infty] & \text{fine index} \\ s \in \mathbb{R} & \text{smoothness} \end{cases}$$

H. Triebel. *Theory of function spaces III*, Birkhäuser, 2006.

Besov spaces as candidates for $\mathcal{X}$

Besov space $B^s_{p,q}$ $\begin{cases} p \in [1,\infty] & \text{integrability} \\ q \in [1,\infty] & \text{fine index} \\ s \in \mathbb{R} & \text{smoothness} \end{cases}$

Norm via Wavelets

$$f(x) = \sum_{(j,m,l)\in I} \underbrace{\langle f, \phi^l_{j,m}\rangle}_{\lambda^l_{j,m}} \phi^l_{j,m}(x)$$

with $(\phi^l_{j,m})_{(j,m,l)\in I}$ a smooth L^2-normalized Daubechies wavelet system.

$$\left\| f \mid B^s_{p,q} \right\| := \left[\sum_{j\in\mathbb{N}_0} \sum_{l=1}^{L_j} 2^{jsq} 2^{jd(\frac{1}{2}-\frac{1}{p})q} \left(\sum_{m\in\mathbb{Z}^d} |\lambda^l_{j,m}|^p \right)^{\frac{q}{p}} \right]^{\frac{1}{q}}$$

H. Triebel. *Theory of function spaces III*, Birkhäuser, 2006.

Properties

- Inclusions: for $s \in \mathbb{R}$, $\varepsilon > 0$ and $1 \leq r \leq q \leq \infty$

$$B^{s+\varepsilon}_{p,q} \subset B^{s+\varepsilon}_{p,\infty} \subset B^{s}_{p,1} \subset B^{s}_{p,r} \subset B^{s}_{p,q}$$

- Dual space for $p, q \neq \infty$:

$$\left(B^{s}_{p,q}\right)^{*} = B^{-s}_{p',q'}$$

Properties

- Inclusions: for $s \in \mathbb{R}$, $\varepsilon > 0$ and $1 \leq r \leq q \leq \infty$

$$B^{s+\varepsilon}_{p,q} \subset B^{s+\varepsilon}_{p,\infty} \subset B^{s}_{p,1} \subset B^{s}_{p,r} \subset B^{s}_{p,q}$$

- Dual space for $p, q \neq \infty$:

$$\left(B^{s}_{p,q}\right)^* = B^{-s}_{p',q'}$$

- Relation to other spaces:

$$\begin{aligned} L^p - \text{spaces} : &\quad B^0_{p,\min\{p,2\}} \subset L^p \subset B^0_{p,\max\{p,2\}} \\ \text{Sobolev spaces} : &\quad W^{s,p} = B^s_{p,p} \quad s \notin \mathbb{Z} \\ \text{Hölder-Zygmund spaces} : &\quad \mathcal{C}^s = B^s_{\infty,\infty} \end{aligned}$$

Properties

- Inclusions: for $s \in \mathbb{R}$, $\varepsilon > 0$ and $1 \leq r \leq q \leq \infty$

$$B^{s+\varepsilon}_{p,q} \subset B^{s+\varepsilon}_{p,\infty} \subset B^{s}_{p,1} \subset B^{s}_{p,r} \subset B^{s}_{p,q}$$

- Dual space for $p, q \neq \infty$:

$$\left(B^{s}_{p,q}\right)^{*} = B^{-s}_{p',q'}$$

- Relation to other spaces:

$$\begin{aligned} L^p - \text{spaces}: &\quad B^0_{p,\min\{p,2\}} \subset L^p \subset B^0_{p,\max\{p,2\}} \\ \text{Sobolev spaces}: &\quad W^{s,p} = B^s_{p,p} \quad s \notin \mathbb{Z} \\ \text{Hölder-Zygmund spaces}: &\quad \mathcal{C}^s = B^s_{\infty,\infty} \end{aligned}$$

- Convexity: with Wavelet-norm

$$B^s_{p,q} \text{ is } \max\{2, p, q\}\text{-convex}$$

K. S. Kazimierski *On the smoothness and convexity of Besov spaces.* **Journal of Inverse and Ill-Posed Problems**, 21:411–429, 2013.

Subgradient Smoothness

Choose $\mathcal{X} = B^0_{p,p}$ for $1 < p \leq 2$

$$\frac{1}{r}\|f \mid \mathcal{X}\|^r = \frac{1}{2}\left[\sum_{j\in\mathbb{N}_0}\sum_{l=1}^{L_j}\sum_{m\in\mathbb{Z}^d}\underbrace{2^{jd(\frac{p}{2}-1)}}_{\to 0,\text{if } p<2}|\lambda^l_{j,m}|^p\right]^{\frac{2}{p}}$$

Subgradient Smoothness

Choose $\mathcal{X} = B^0_{p,p}$ for $1 < p \leq 2$

$$\frac{1}{r}\|f \mid \mathcal{X}\|^r = \frac{1}{2}\left[\sum_{j\in\mathbb{N}_0}\sum_{l=1}^{L_j}\sum_{m\in\mathbb{Z}^d}\underbrace{2^{jd(\frac{p}{2}-1)}}_{\to 0,\text{if } p<2}|\lambda^l_{j,m}|^p\right]^{\frac{2}{p}}$$

If $f^\dagger$ is in a smooth subspace of $B^0_{p,p}$, is there a smooth subspace of $B^0_{p',p'}$ containing f^*?

Subgradient Smoothness

Choose $\mathcal{X} = B^0_{p,p}$ for $1 < p \leq 2$

$$\frac{1}{r}\|f \mid \mathcal{X}\|^r = \frac{1}{2}\left[\sum_{j\in\mathbb{N}_0}\sum_{l=1}^{L_j}\sum_{m\in\mathbb{Z}^d}\underbrace{2^{jd(\frac{p}{2}-1)}}_{\to 0,\text{if } p<2}|\lambda^l_{j,m}|^p\right]^{\frac{2}{p}}$$

If $f^\dagger$ is in a smooth subspace of $B^0_{p,p}$, is there a smooth subspace of $B^0_{p',p'}$ containing f^*?

Theorem

Let $f^* \in \partial\frac{1}{2}\|f^\dagger \mid B^0_{p,p}\|^2$ *then*

$$f^\dagger \in B^s_{p,\infty} \text{ for } s > 0 \quad \Longleftrightarrow \quad f^* \in B^{s(p-1)}_{p',\infty}.$$

Proof.

$$f^* = \sum_{(j,m,l)\in I}\mu^l_{j,m}\phi^l_{j,m}(x) \text{ with } \mu^l_{j,m} = \|f^\dagger \mid B^0_{p,p}\|^{2-p}2^{jd(\frac{p}{2}-1)}\frac{\lambda^l_{j,m}}{|\lambda^l_{j,m}|^{2-p}}.$$

Tikhonov functional

$$T_{\alpha,g^\delta}(q) = \frac{1}{2\alpha}\left\| F(q) - g^\delta \mid \mathcal{Y} \right\|^2 + \frac{1}{2}\left\| q \mid B^0_{p,p} \right\|^2 + \iota_{\{\Im(\cdot)\geq 0, \mathrm{supp}(\cdot)\subset B(r)\}}(q) + \iota_{\{\|\cdot \mid L^\infty\| \leq C_\infty\}}(q)$$

Needed for ill-posedness inequality ($\rightsquigarrow$ CGO-solultions)

Tikhonov functional

$$T_{\alpha,g^\delta}(q) = \frac{1}{2\alpha}\left\|F(q) - g^\delta \mid \mathcal{Y}\right\|^2 + \frac{1}{2}\left\|q \mid B^0_{p,p}\right\|^2$$
$$+ \iota_{\{\Im(\cdot)\geq 0, \mathrm{supp}(\cdot)\subset B(r)\}}(q) + \iota_{\{\|\cdot \mid L^\infty\| \leq C_\infty\}}(q)$$

Needed for ill-posedness inequality ($\rightsquigarrow$ CGO-solultions)

Bonus: Weak topologies of $B^0_{p,p}$ and L^2 conincide on $\mathrm{dom}(T_{\alpha,\cdot})$.

Tikhonov functional

$$T_{\alpha,g^\delta}(q) = \frac{1}{2\alpha}\left\|F(q) - g^\delta \mid \mathcal{Y}\right\|^2 + \frac{1}{2}\left\|q \mid B^0_{p,p}\right\|^2 + \iota_{\{\Im(\cdot)\geq 0, \mathrm{supp}(\cdot)\subset B(r)\}}(q) + \iota_{\{\|\cdot \mid L^\infty\|\leq C_\infty\}}(q)$$

Needed for ill-posedness inequality ($\rightsquigarrow$ CGO-solultions)

Bonus: Weak topologies of $B^0_{p,p}$ and L^2 conincide on $\mathrm{dom}(T_{\alpha,\cdot})$.

Theorem

This strategy is regularizing.

Directly apply results of:

A. Lechleiter, K. S. Kazimierski and M. Karamehmedović *Tikhonov regularization in L^p applied to inverse medium scattering.* **Inverse Problems**, 29:075003, 2013.

Main result

Theorem (Variational Source Condition)

$R > r > 0$, $E \geq 1$, $C_\infty > 0$, $2 \geq p > 1$, $s > 0$ and $C_s > 0$.
Let the true potential $q^\dagger$ satisfy:

$$\operatorname{supp}(q^\dagger) \subset B(r), \quad \Im(q^\dagger) \geq 0, \quad \|q^\dagger \mid L^\infty\| \leq C_\infty, \quad \|q^\dagger \mid B^s_{p,\infty}\| \leq C_s$$

Then $\exists c > 0$ such that for all $q \in \operatorname{dom}(T_{\alpha,\cdot})$ the VSC

$$\langle q^*, q^\dagger - q \rangle \leq \frac{1}{2}\Delta_{\frac{1}{2}\|\cdot \mid B^0_{p,p}\|^2}(q, q^\dagger) + cC_s(1 + C_s)\left(E^3\delta^{\frac{1}{2}} + (1 + C_\infty^2)\left(E + \ln^2(3 + \delta^{-2})\right)^{-\mu}\right)$$

holds true, where

$$\delta := \|F(q) - F(q^\dagger) \mid L^2\|, \qquad \mu = \min\left\{\frac{2}{4-p}, s(p-1)\right\}.$$

Corollaries

Corollary (Convergence rate)

Let q_α^δ be the solution of the Tikhonov functional for optimal α, then

$$\left\|q^\dagger - q_\alpha^\delta \mid B^0_{p,p}\right\| \lesssim (1+C_s)\left(E^3\delta^{\frac{1}{2}} + \left(1 + C_\infty^2\right)\left(E + \ln^2(3+\delta^{-2})\right)^{-\mu}\right)^{1/2}$$

Corollaries

Corollary (Convergence rate)

Let q_α^δ *be the solution of the Tikhonov functional for optimal* α, *then*

$$\left\| q^\dagger - q_\alpha^\delta \mid B_{p,p}^0 \right\| \lessapprox (1+C_s)\left(E^3 \delta^{\frac{1}{2}} + \left(1 + C_\infty^2\right)\left(E + \ln^2(3+\delta^{-2})\right)^{-\mu}\right)^{1/2}$$

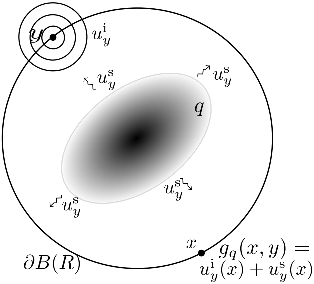
Corollary (Stability)

Let q_1, q_2 *fulfill the conditions on* $q^\dagger$, *then*

$$\left\| q_1 - q_2 \mid B_{p,p}^0 \right\| \lessapprox (1+C_s)\left(E^3 \delta^{\frac{1}{2}} + \left(1 + C_\infty^2\right)\left(E + \ln^2(3+\delta^{-2})\right)^{-\mu}\right)^{1/2}$$

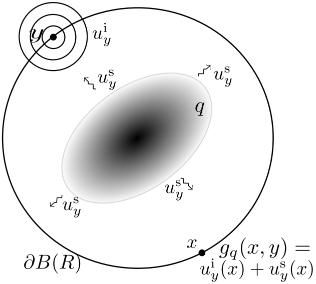
Summary

Method to recover q from g



Summary

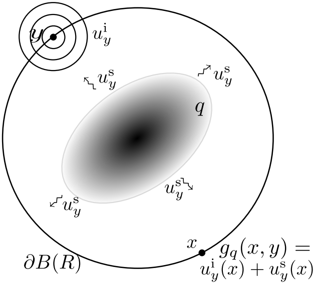
Method to recover q from g



- ✓ Regularization strategy
  - ✓ no smoothness enforcement
  - ✓ sparsity enforcing

Summary

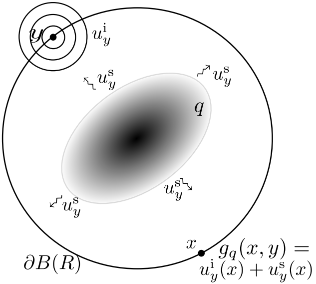
Method to recover q from g



- ✓ Regularization strategy
  - ✓ no smoothness enforcement
  - ✓ sparsity enforcing
- ✓ Proved rates of convergence
  - ✓ Hölder-logarithmic w.r.t. energy E
  - ✗ Unbounded exponent

Summary

Method to recover q from g

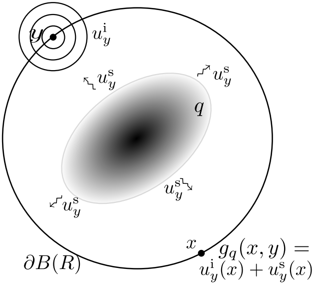


- ✓ Regularization strategy
  - ✓ no smoothness enforcement
  - ✓ sparsity enforcing
- ✓ Proved rates of convergence
  - ✓ Hölder-logarithmic w.r.t. energy E
  - ✗ Unbounded exponent

Prize: Add $\iota_{\{\|\cdot\,|\,L^\infty\| \leq C_\infty\}}(q)$

Summary

Method to recover q from g



- ✓ Regularization strategy
  - ✓ no smoothness enforcement
  - ✓ sparsity enforcing
- ✓ Proved rates of convergence
  - ✓ Hölder-logarithmic w.r.t. energy E
  - ✗ Unbounded exponent

Prize: Add $\iota_{\{\|\cdot\|_{L^\infty}\| \leq C_\infty\}}(q)$

Thank you for your attention!