Verification of a Variational Source Condition for Inverse Medium Scattering

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Recent Developments in Inverse Problems
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Outline

Tikhonov regularization

General Setup

- Let $\mathbb{X}, \mathbb{Y}$ be Banach spaces
- $F\colon \operatorname{dom}(F) \subset \mathbb{X} \to \mathbb{Y}$ be a (possibly nonlinear) forward operator
- $f^\dagger \in \operatorname{dom}(F)$ the true solution
- $g^\delta \in \mathbb{Y}$ observed data with $\|g^\delta - F(f^\dagger)\|_{\mathbb{Y}} \leq \delta$

Tikhonov regularization: Find an approximate solution

$$f_\alpha^\delta \in \operatorname*{arg\,min}_{f \in \operatorname{dom}(F)} \left[\frac{1}{\alpha} \left\| F(f) - g^\delta \right\|_{\mathbb{Y}}^2 + \Omega(f) \right],$$

where Ω is an appropriate penalty term.

Distance to true solution

Spectral source conditions:

$$f^\dagger = \varphi\left(F'[f^\dagger]^* F'[f^\dagger]\right)\omega$$

with an index function φ, $\rightsquigarrow$ additional restrictive requirements for nonlinear operators (tangential cone condition)

H. Engl, M. Hanke and A. Neubauer. *Regularization of inverse problems*, Kluwer, 1996.

Variational source conditions (VSC):

$$\forall f \in \mathrm{dom}(F): \quad \beta\Delta_\Omega(f, f^\dagger) \leq \Omega(f) - \Omega(f^\dagger) + \psi\left(\left\|F(f) - F(f^\dagger)\right\|_{\mathbb{Y}}^2\right)$$

for a concave index function ψ and a $\beta \in (0,1]$.
First used (with $\psi(t) = c\sqrt{t}$) in

B. Hofmann, B. Kaltenbacher, C. Pöschl, and O. Scherzer. *A convergence rates result for Tikhonov regularization in Banach spaces with non-smooth operators*. **Inverse Problems** 23:987–1010, 2007.

Advantages of VCSs

- simplify proofs, e.g. one can easily show that a VSC implies the convergence rate
$$\beta\Delta_\Omega(f_\alpha^\delta, f^\dagger) \leq 4\psi(\delta^2)$$
for the optimal choice of the regularization parameter α
  - M. Grasmair. *Generalized Bregman distances and convergence rates for non-convex regularization methods.* **Inverse Problems** 26:115014, 2010.
  - F. Werner and T. Hohage. *Convergence rates in expectation for Tikhonov-type regularization of inverse problems with Poisson data.* **Inverse Problems** 28:104004, 2012.
- for linear operators between Hilbert space even necessary conditions for certain rates of convergence
  - J. Flemming, B. Hofmann, and P. Mathé. *Sharp converse results for the regularization error using distance functions.* **Inverse Problems**, 27:025006, 2011.
- no differentiability assumption $\rightsquigarrow$ no restrictive assumption connecting F and F' needed (tangential cone condition)
- allow extension to Banach spaces and general data misfit/penalty terms

...but

Verification of VSCs:

- Reformulations of VSC with $\psi(x) = \sqrt{x}$ for a phase retrieval and an option pricing problem.

B. Hofmann, B. Kaltenbacher, C. Pöschl, and O. Scherzer. *A convergence rates result for Tikhonov regularization in Banach spaces with non-smooth operators*. **Inverse Problems** 23:987–1010, 2007.

- Spectral source conditions imply VSCs.
- For linear operators F with ℓ^q penalty term via the range of F^*

M. Burger, J. Flemming, and B. Hofmann. *Convergence rates in ℓ^1-regularization if the sparsity assumption fails.* **Inverse Problems**, 29:025013, 2013.

S. W. Anzengruber, B. Hofmann, and R. Ramlau. *On the interplay of basis smoothness and specific range conditions occurring in sparsity regularization.* **Inverse Problems**, 29:125002, 2013.

J. Flemming and M. Hegland. *Convergence rates in ℓ^1-regularization when the basis is not smooth enough.* **Applicable Analysis**, 94:464–476, 2015.

$\rightsquigarrow$ few verifications so far

VSC vs. stability estimates

Let $K \subset \mathrm{dom}(F)$ be some smoothness class (e.g. a Sobolev ball), and Δ a symmetric error measure.

Variational source condition: $\forall f^\dagger \in K, f \in \mathrm{dom}(F)$:

$$\beta\Delta(f, f^\dagger) \leq \Omega(f) - \Omega(f^\dagger) + \psi\left(\|F(f) - F(f^\dagger)\|_{\mathbb{Y}}^2\right)$$

Conditional stability estimate: $\forall f_1, f_2 \in K$:

$$\beta\Delta(f_1, f_2) \leq \qquad\qquad \psi\left(\|F(f_1) - F(f_2)\|_{\mathbb{Y}}^2\right)$$

VSC $\Longrightarrow$ Stability:

- W.l.o.g. $\Omega(f_1) \geq \Omega(f_2)$, choose $f_1 = f^\dagger$, $f_2 = f$

Stability $\overset{???}{\Longrightarrow}$ VSC:

- sign of $\Omega(f) - \Omega(f^\dagger)$ unknown
- VSC must hold on the larger set $\mathrm{dom}(F)$

Forward medium scattering

Given a refractive index n and one/several initial wave(s) u^{i} fulfilling the Helmholtz equation $\Delta u^{\mathrm{i}} + \kappa^2 u^{\mathrm{i}} = 0$ we want to find the total field(s) $u = u^{\mathrm{i}} + u^{\mathrm{s}}$ solving

$$\begin{aligned} \Delta u + \kappa^2 n u &= 0 && \text{in } \mathbb{R}^3, \\ \frac{\partial u^{\mathrm{s}}}{\partial r} - i\kappa u^{\mathrm{s}} &= \mathcal{O}\left(r^{-2}\right) && \text{as } r = |x| \to \infty. \end{aligned}$$

for a fixed wave number κ.

Assumption on the contrast $f = 1 - n$

$$f \in \mathfrak{D} := \left\{ f \in L^\infty(\mathbb{R}^3) \colon \operatorname{supp}(f) \subset B(\pi), \Re(f) \leq 1, \Im(f) \leq 0 \right\}$$

with $B(R) := \{x \in \mathbb{R}^3 \colon |x| \leq R\}$.

Near field inverse problem

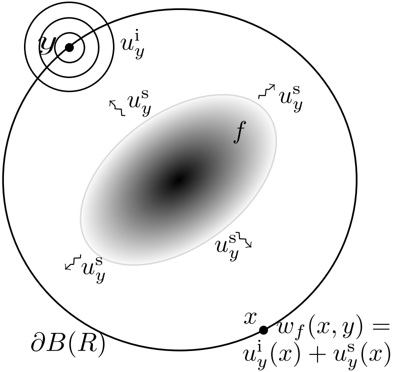
Incident fields are point source waves

$$u_y^{\mathrm{i}}(x) = \frac{1}{4\pi}\frac{\mathrm{e}^{i\kappa|x-y|}}{|x-y|}$$

for all $y \in \partial B(R)$ with $R > \pi$.
Measurements are the total fields

$$w_f(x,y) = u_y^{\mathrm{i}}(x) + u_y^{\mathrm{s}}(x)$$

on $\partial B(R) \times \partial B(R)$.
Functional Setting:



$$F_{\mathrm{near}} : \mathfrak{D} \cap H_0^m(B(\pi)) \to L^2\left(\partial B(R) \times \partial B(R)\right), \quad f \mapsto w_f$$

We choose $\mathbb{X} = H_0^m(B(\pi))$ for $m > 3/2$ and $\Omega = \frac{1}{2}\|\cdot\|_{H^m}^2$.

Main theorem

Theorem

Assume that $3/2 < m < s$, $s \neq 2m + 3/2$ *and* $f^\dagger \in \mathfrak{D}$ *satisfies* $\|f^\dagger\|_{H^s} \leq C_s$ *for some* $C_s \geq 0$. *Then a VSC holds true for the operator* F_{n} *with* $\beta = 1/2$, *and* ψ *given by*

$$\psi_{\mathrm{near}}(t) := A\left(\ln(3 + t^{-1})\right)^{-2\mu}, \qquad \mu := \min\left\{1, \frac{s-m}{m+3/2}\right\},$$

where the constant $A > 0$ *depends only on* m, s, C_s, κ *and* R.

Remark: Theorem holds true for any $\beta \in (0,1)$, but ψ_{near} depends on β.

T. Hohage, F. Weidling. *Verification of a variational source condition for acoustic inverse medium scattering problems.* **Inverse Problems**. 31:075006, 2015.

Corollaries

Corollary (convergence rate)

Under the assumptions of the previous theorem the error bound

$$\left\|f_\alpha^\delta - f^\dagger\right\|_{H^m} \leq 2\sqrt{2A}\left(\ln(3+\delta^{-2})\right)^{-\mu}$$

holds true for Tikhonov regularization for optimal α.

Corollary (stability estimate)

Suppose $\frac{3}{2} < m < s$, $s \neq 2m + 3/2$ *and* f_1 *and* f_2 *satisfy* $f_j \in \mathfrak{D}$ *with* $\|f_j\|_{H^s} \leq C_s$ *for* $j = 1, 2$ *and some* $C_s > 0$. *Then*

$$\|f_1 - f_2\|_{H^m} \leq \sqrt{2A}\left[\ln\left(3 + \|F_{\mathrm{near}}(f_1) - F_{\mathrm{near}}(f_2)\|_{L^2(\partial B(R)^2)}^{-2}\right)\right]^{-\mu}.$$

Known results (... very incomplete list)

Instability results

show that logarithmic estimates are essentially optimal

N. Mandache. *Exponential instability in an inverse problem for the Schrödinger equation.* **Inverse Problems**, 17:1435–1444, 2001.

Stability results

local stability, strong norm in image space, unknown coefficient $\mu \leq 1$:

P. Stefanov. *Stability of the inverse problem in potential scattering at fixed energy.* **Ann. Inst. Fourier (Grenoble)**, 40:867–884, 1990.

global stability, $\mathbb{Y} = L^2$ and explicit $\mu \leq 1$:

P. Hähner and T. Hohage. *N*ew stability estimates for the inverse acoustic inhomogeneous medium problem and applications. **SIAM J. Math. Anal.**, 33:670–685, 2001.

explicit dependence of κ yielding asymptotically Hölder stability and $\mu \to \infty$ as $s \to \infty$

M. I. Isaev and R. G. Novikov. *Effectivized Hölder-logarithmic stability estimates for the Gel'fand inverse problem.* **Inverse Problems**, 30:095006, 2014.

Reformulation

Choice of norm on $H_0^m(B(\pi))$:

$$\|f\|_{H^m}^2 := \sum_{\gamma \in \mathbb{Z}^3} (1+|\gamma|^2)^m \left|\widehat{f}(\gamma)\right|^2$$

where $\widehat{f}(\gamma) = (2\pi)^{-3/2} \int_{C(\pi)} f(x) \mathrm{e}^{-i\gamma\cdot x} \mathrm{d}x$ are the Fourier coefficients of f in the cube $C(\pi) = (-\pi, \pi)^3$ with f extended by 0 outside of $B(\pi)$.

Equivalent formulation of the variational source condition:

$$\Re \sum_{\gamma \in \mathbb{Z}^3} (1+|\gamma|^2)^m \widehat{f^\dagger}(\gamma) \overline{\left(\widehat{f^\dagger}(\gamma) - \widehat{f}(\gamma)\right)} = \Re \left\langle f^\dagger, f^\dagger - f \right\rangle_{H^m}$$

$$\leq \frac{1-\beta}{2} \left\|f^\dagger - f\right\|_{H^m}^2 + \psi\left(\left\|F_{\mathrm{near}}(f) - F_{\mathrm{near}}(f^\dagger)\right\|_{\mathbb{Y}}^2\right)$$

The simple part...

A universal bound: Assume that $\|f^\dagger - f\|_{H^m} \geq 4C_s$, then

$$\Re \left\langle f^\dagger, f^\dagger - f \right\rangle_{H^m} \leq \left\|f^\dagger\right\|_{H^m} \left\|f^\dagger - f\right\|_{H^m} \leq C_s \left\|f^\dagger - f\right\|_{H^m} \leq \frac{1}{4} \left\|f^\dagger - f\right\|_{H^m}^2$$

$\rightsquigarrow$ VSC independent of δ

Bound on high frequencies: Let $\varrho > 0$, then

$$\Re \sum_{\gamma \in \mathbb{Z}^3 \setminus B(\varrho)} (1 + |\gamma|^2)^m \widehat{f^\dagger}(\gamma) \overline{\left(\widehat{f^\dagger}(\gamma) - \widehat{f}(\gamma)\right)} \leq \frac{1}{8} \left\|f^\dagger - f\right\|_{H^m}^2 + 2C_s \varrho^{2(m-s)}$$

by Cauchy-Schwarz's and Young's inequality.

Low frequencies

Lemma

Let $\pi < R < R'$. Then there exists a positive constant C such that for all contrasts $f_1, f_2 \in \mathfrak{D}$ with corresponding near fields w_1 and w_2, and for all solutions $u_j \in H^2(B(R'))$ to $\Delta u_j + \kappa^2 u_j = \kappa^2 f_j u_j$ the following estimate holds true:

$$\left| \int_{B(\pi)} (f_1 - f_2)\, u_1 u_2 \mathrm{d}x \right| \leq C \, \|w_1 - w_2\|_{L^2(\partial B(R)^2)} \, \|u_1\|_{L^2(B(R'))} \, \|u_2\|_{L^2(B(R'))} \, .$$

P. Hähner and T. Hohage. New stability estimates for the inverse acoustic inhomogeneous medium problem and applications. **SIAM J. Math. Anal.**, 33:670–685, 2001.

Note: If we could choose $u_j = \exp(i\zeta_j \cdot x)$ with $\zeta_j \in \mathbb{C}^3$, $\zeta_1 = \overline{\zeta_2}$ and $\zeta_j \cdot \zeta_j = \sum_{l=1}^3 \zeta_{j,l}^2 = \kappa^2$ we would get bounds on Fourier coefficients of $f_1 - f_2$.

CGO solutions

Solutions to $\Delta u + \kappa^2 u = \kappa^2 f u$ of the form

$$u(x) = e^{i\zeta\cdot x}(1 + v(x)) \quad \text{with } \zeta \in \mathbb{C}^3, \zeta_1 \cdot \zeta_2 = \sum_{j=1}^{3} \zeta_j^2 = \kappa^2, t := |\Im(\zeta)| > 0$$

with "small" v are called Complex Geometric Optics solutions. For existence and estimates

P. Hähner. *A* Periodic Faddeev-Type Solution Operator. **Journal of Differential Equations**, 128:300–308, 1996.

Lemma

Let $C_m > 0$, $m > 3/2$, $\pi < R < R'$ and f_1 and f_2 be contrasts with $f_j \in \mathfrak{D}$ and $\|f_j\|_{H^m} \leq C_m$ with corresponding near field data w_j for $j = 1, 2$. Let $t \geq t_0(\kappa, C_m, \ldots)$ and $1 \leq \varrho \leq 2\sqrt{\kappa^2 + t^2}$. Then there exists a constant $C > 0$ such that for all $\gamma \in \mathbb{Z}^3 \cap B(\varrho)$ we have

$$\left|\widehat{f_1}(\gamma) - \widehat{f_2}(\gamma)\right| \leq C\mathrm{e}^{4R't} \|w_1 - w_2\|_{L^2(\partial B(R)^2)} + \frac{C}{t} \|f_1 - f_2\|_{H^m}.$$

Conclusion

- First rigorous verification of a variational source condition for a nonlinear inverse problem in PDEs
- First rigorous proof of (logarithmic) convergence rates for Tikhonov applied to medium scattering under Sobolev smoothness
- Proof uses techniques from stability estimates

but ...

- We cannot guarantee that a global minimum of the Tikhonov functional can be computed. This still requires F' and conditions such as the tangential cone condition.

Outlook

- other medium scattering problems, in particular electromagnetic
- wave number dependence, Hölder-logarithmic VSC
- exponents $\mu > 1$
- Banach norm penalty terms Ω
- Which other stability estimates can be sharpened to variational source conditions?

Thank you for your attention