Generalized SART-methods for robust and efficient tomography

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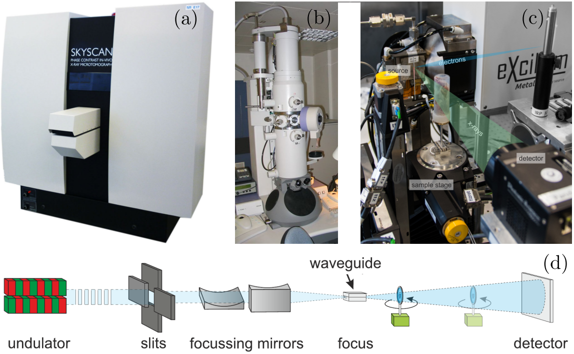
100 Years of the Radon Transform
Minisymposium MS 15: Towards Robust Tomography

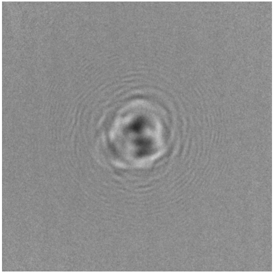
March 31st, 2017

Outline

Motivation

Imaging Model





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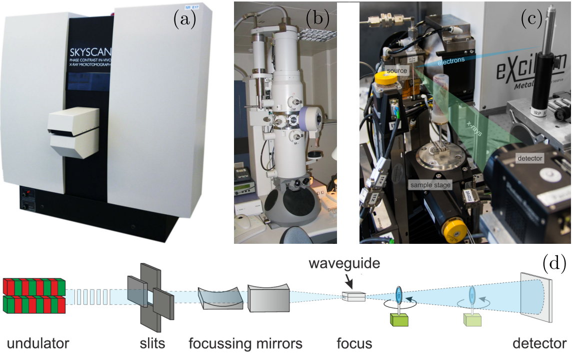

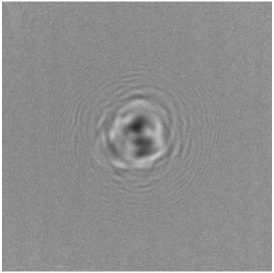

***Semidiscrete* model:** parallel-/cone-projectors $\mathscr{P}_j$, image formation F_j

$$\underbrace{g^{\mathrm{obs}}}_{\text{data}} = \begin{pmatrix} g_1^{\mathrm{obs}} \\ \vdots \\ g_{N_{\mathrm{proj}}}^{\mathrm{obs}} \end{pmatrix} = \begin{pmatrix} F_1 \circ \mathscr{P}_1 \\ \vdots \\ F_{N_{\mathrm{proj}}} \circ \mathscr{P}_{N_{\mathrm{proj}}} \end{pmatrix} \underbrace{(f^\dagger)}_{\text{object}} + \underbrace{\boldsymbol{\delta}}_{\text{errors}}, \qquad \mathscr{P}_j(f) = \begin{cases} \int_{\mathbb{R}} f(\mathbf{x}_\perp + z\boldsymbol{\theta}_j)\mathrm{d}z \\ \int_0^\infty f(\mathbf{s}_j + t\boldsymbol{\varphi})\mathrm{d}t \end{cases}$$

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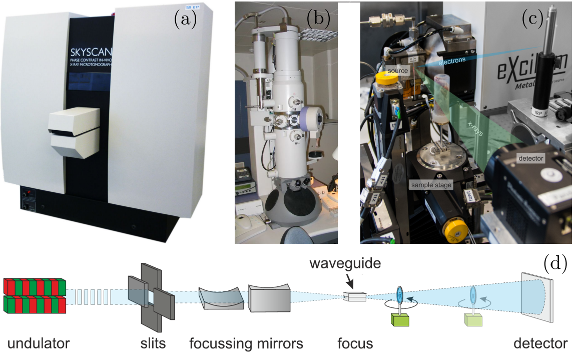

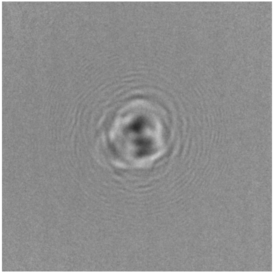

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(*No* assumptions on incident angles $\boldsymbol{\theta}_j$ / source positions $\boldsymbol{s}_j$!)

Reconstruction methods: Efficiency vs. Flexibility

Tomographic reconstruction problem

Recover a 3D-object f from measured data $g^{\text{obs}} = F(\mathscr{P}(f)) + \delta$

- Numerical dimensions ($N \sim 10^3$): object f sampled on $\mathcal{O}(N^3)$ voxels, g^{obs} composed of $N_{\text{proj}} \sim N$ images of size $\mathcal{O}(N^2)$
- Complexity: $f \mapsto \left(\mathscr{P}_1(f), \ldots, \mathscr{P}_{N_{\text{proj}}}(f)\right) \rightsquigarrow \mathcal{O}(N_{\text{proj}} N^3) \sim \mathcal{O}(N^4)$ FLOPs

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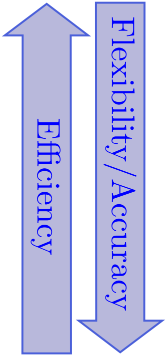

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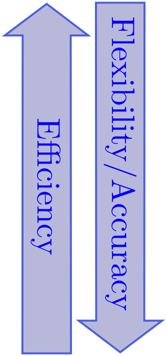

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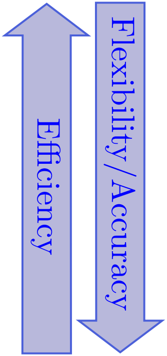

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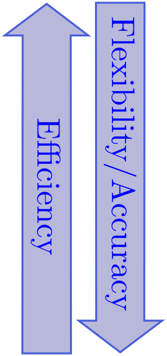
$$f^{\text{rec}} = \operatorname*{argmin}_{f \in X} \underbrace{\mathcal{S}(g^{\text{obs}}; F \circ \mathscr{P}(f))}_{\substack{\text{data-fidelity: } L^1, L^2, \\ \text{Kullback-Leibler}}} + \underbrace{\mathcal{R}(f)}_{\substack{\text{regularization} \\ + \text{ constraints}}}$$



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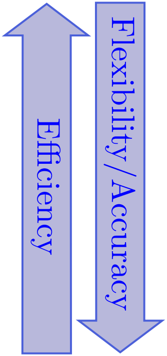

SART and its Generalization

Classical Kaczmarz-iterations and SART

Classical Kaczmarz-iterations:

$$f_{k+1} = \underset{f \in L^2(\Omega)}{\operatorname{argmin}} \left\{ \|f - f_k\|_{L^2}^2 \text{ s.t. } \mathscr{P}_{j_k}(f) = g_{j_k}^{\text{obs}} \right\}$$

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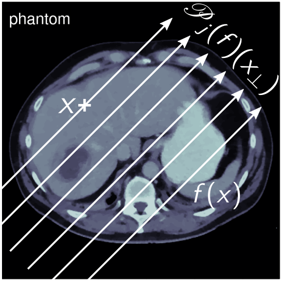

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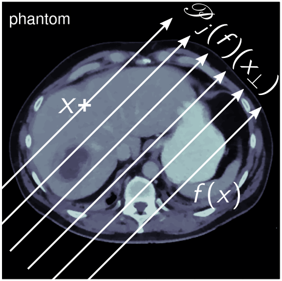

Discretize (CK) $\rightsquigarrow$ Simultaneous ART [1]:

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SART-iterates are *not* minimizers of a discrete optimization problem.

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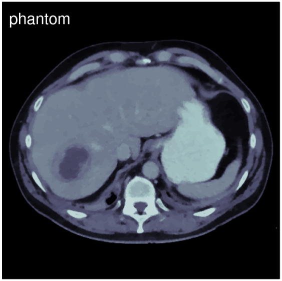

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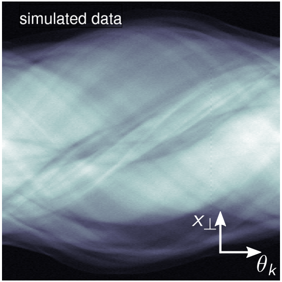

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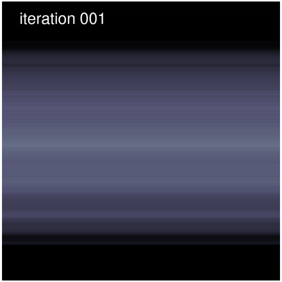

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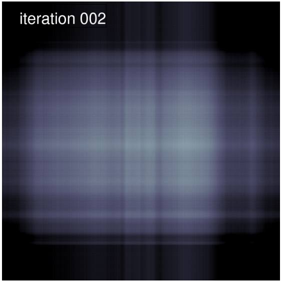

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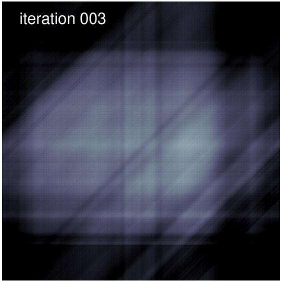

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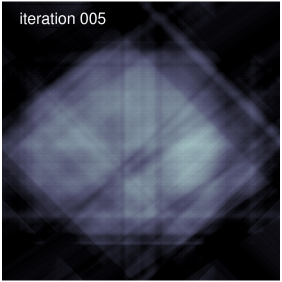

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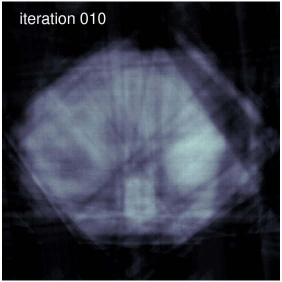

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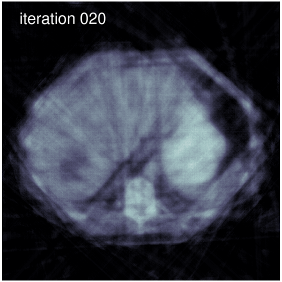

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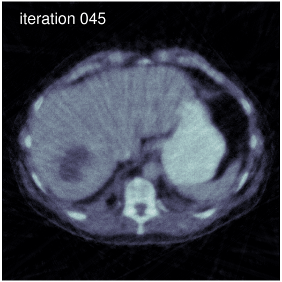

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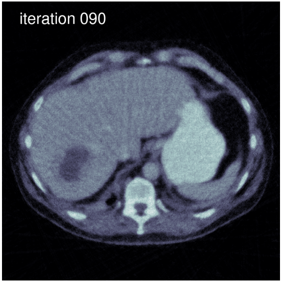

Discretize (CK) $\rightsquigarrow$ Simultaneous ART [1]:

✓ Fast semi-convergence in $\mathcal{O}(1)$ cycles

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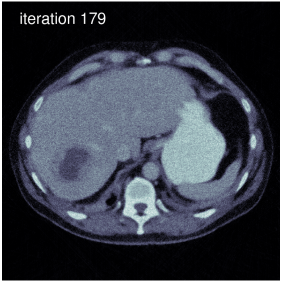

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- ✓ Fast semi-convergence in $\mathcal{O}(1)$ cycles
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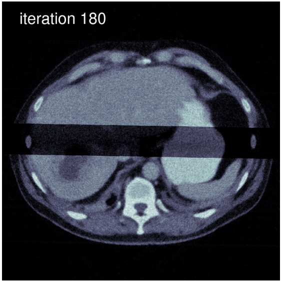

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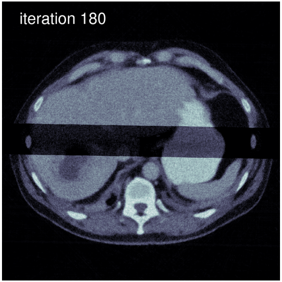

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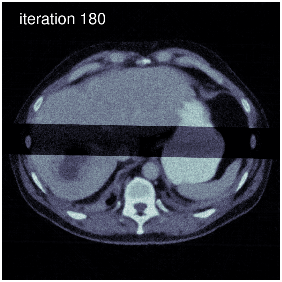

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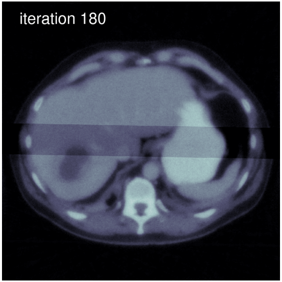

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$$f^{rec} = \operatorname{argmin}_{f \in X} \mathcal{S}(g^{obs}; F(\mathscr{P}(f))) + \mathcal{R}(f) \qquad \text{(genTikhonov)}$$

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Generalized SART: Compute (genKaczmarz) via SART-like scheme

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Generalized SART-Theorem

Theorem 1 (Generalized SART-scheme)

Let $\mathscr{P} : X \to Y$ be linear-bounded on Hilbert spaces X, Y s.t. $\mathscr{P}^*$ has closed range. Let $\mathcal{R} : X \to \bar{\mathbb{R}}$, $\tilde{\mathcal{S}} : Y \to \bar{\mathbb{R}}$ be functionals. Let

$$f^\dagger \in \operatorname*{argmin}_{f \in X} \left(\tilde{\mathcal{S}}\left(\mathscr{P}(f)\right) + \mathcal{R}(f) \right).$$

Moreover, for some $f^{\mathrm{ref}} \in X$, let $\mathcal{R}$ satisfy for all $p \in Y$, $f_0 \in kern(\mathscr{P})$

$$\mathcal{R}(f^{\mathrm{ref}} + \mathscr{P}^*(p) + f_0) \geq \mathcal{R}(f^{\mathrm{ref}} + \mathscr{P}^*(p)), \tag{A1}$$

Then there exists a (possibly distinct) global minimizer $\tilde{f}^\dagger$ given by

$$p^{\mathrm{ref}} = \mathscr{P}(f^{\mathrm{ref}}) \tag{genSART-1}$$

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Points of consideration:

1. Are the assumptions met for tomographic Kaczmarz-iterations?
2. Is the optimization in (genSART-2) sufficiently easy?

Application to Kaczmarz-method

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- ✓ If $\mathscr{P} = \mathscr{P}_{j_k} : L^2(\Omega) \to Y$ with $Y := L^2_{u_{\mathscr{P}}^{-1}}(D)$, then $\mathscr{P}^*$ has closed range and $\mathscr{P}\mathscr{P}^* = \mathrm{id}_Y$ is trivial
- ✓ Practically *no* restrictions on $\tilde{\mathcal{S}}(p) := \mathcal{S}_{j_k}(g_{j_k}^{\mathrm{obs}}; F_{j_k}(p))$
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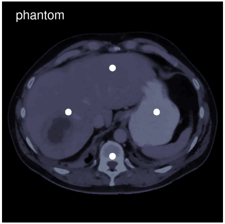
- Similar results for L^q-, weighted L^2- (and gradient-L^q-)penalties

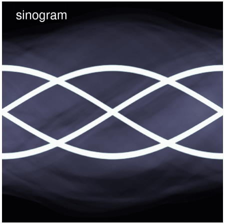
Applications in Robust Reconstruction

Robust SART

Robust tomographic reconstruction problem:

Heavy metal inclusions in CT-scanned objects result in large outliers
$\rightsquigarrow$ highly non-Gaussian data error statistics





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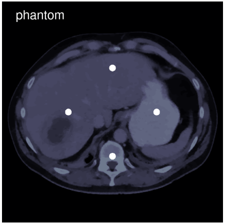

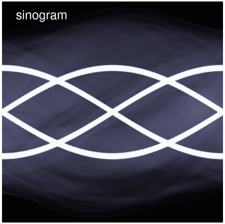

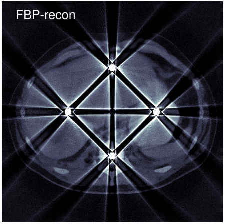

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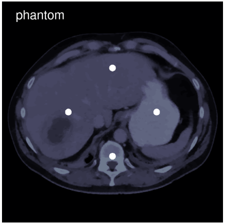

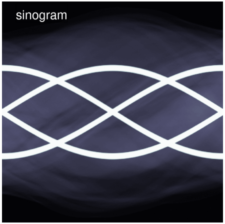

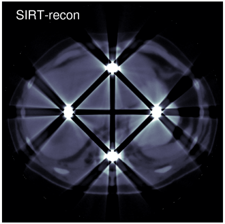

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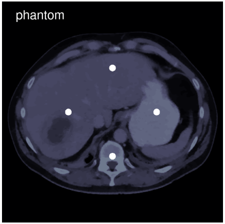

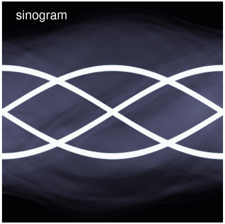

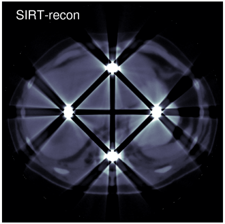

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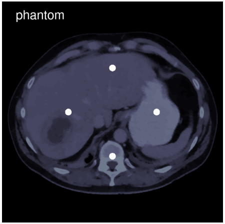

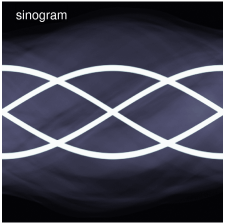

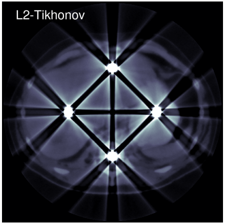

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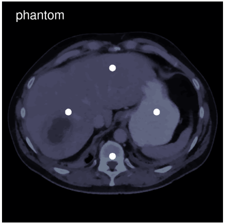

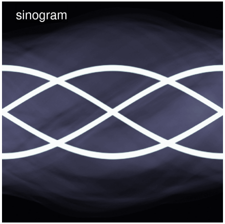

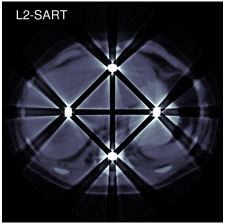

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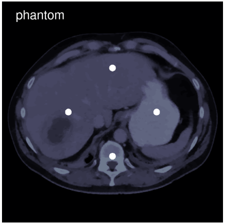

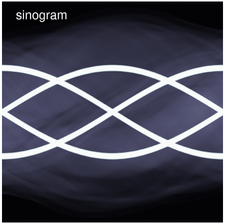

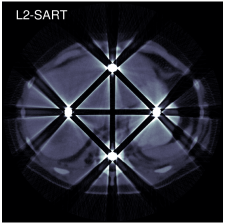

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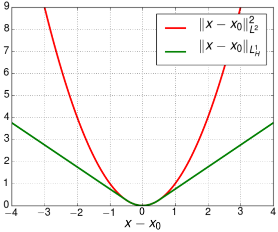

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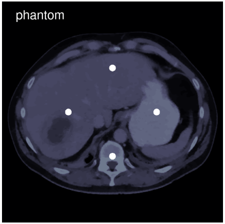

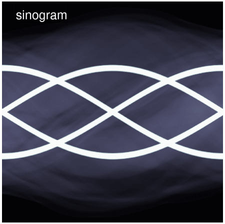

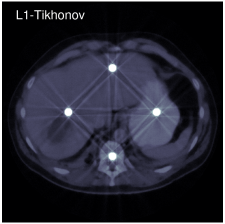

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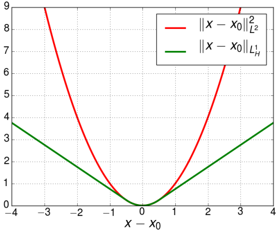

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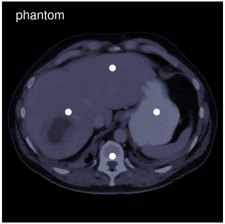

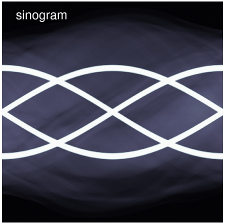

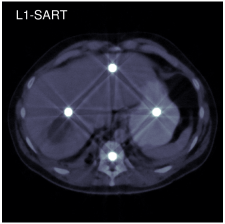

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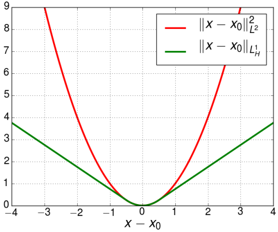

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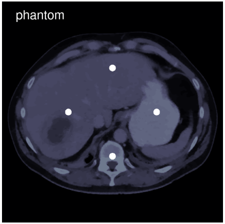

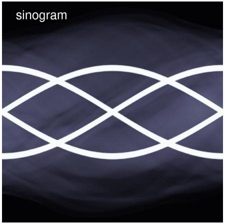

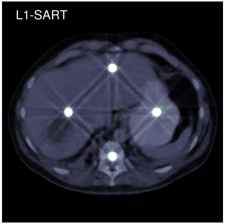

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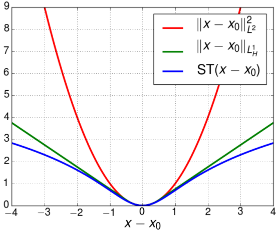

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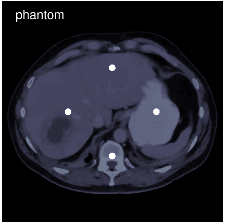

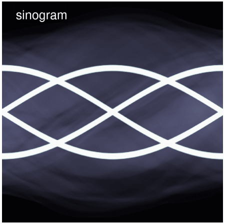

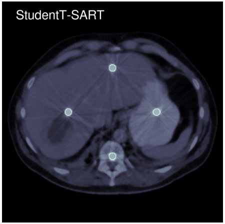

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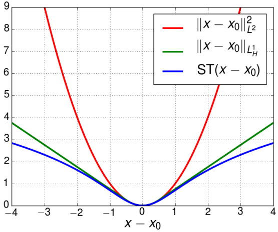

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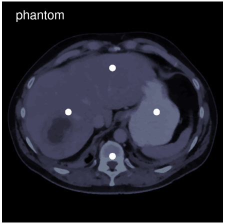

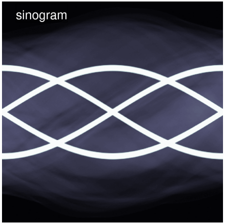

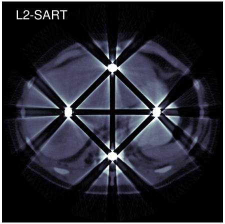

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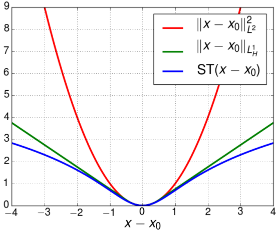

Robust Tikhonov vs Robust SART

Student-T-Tikhonov [3]:

$$f^{\text{Tikh}} = \text{argmin}_f \{\text{ST}_\nu(\mathscr{P}(f) - g^{\text{obs}}) + \alpha ||f||^2_{L^2}\}$$

$$\text{ST}_\nu(p) = \int \nu \log\left(1 + p(x)^2/\nu\right) \mathrm{d}x$$

- Large-scale non-convex optimization problem
- No analytical solution
- *Algorithm* [3]: Iterative solution of weighted least-squares problems
- *Complexity:* $\mathcal{O}(100)$ evaluations of $\mathscr{P}$, $\mathscr{P}^*$

[3] F. Bleichrodt, T. van Leeuwen, K.J. Batenburg (2015). *Fully 3D Conf. Proc.*

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- Non-convex optimization problems
- Generalized SART-scheme:
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Computed Tomography with Poisson Noise

Monochromatic CT-model:

$$g_j = \underbrace{I_0}_{\text{illumination}} \cdot \exp\left(-\mathscr{P}_j(f)\right) = F_j(\mathscr{P}_j(f))$$

$$g_{ji}^{\text{obs}} \sim \text{Poi}\left(\int_{D_i} g_j \, \mathrm{d}x\right)$$

- Ideal data given by Beer-Lambert's law
- M pixels D_i count number of incident photons $\rightsquigarrow$ Poisson-random variables [5]

[5] T. Hohage and F. Werner (2016). *Inverse Problems* **32**(9):093001.

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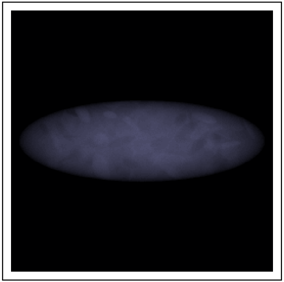

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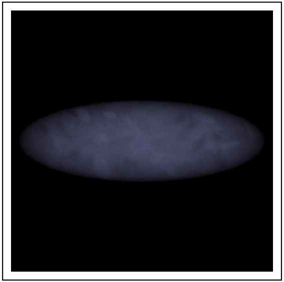
$$g_{ji}^{\text{obs}} \sim \text{Poi}\left(\int_{D_i} g_j \, dx\right)$$

- Ideal data given by Beer-Lambert's law
- M pixels D_i count number of incident photons $\rightsquigarrow$ Poisson-random variables [5]
- Non-uniform noise level $\sim g_j^{-1/2}$ in logarithmized data $p_j^{\text{obs}} = -\ln(g_j^{\text{obs}}/I_0)$



Negative log-likelihood: $\rightsquigarrow$ *Kullback-Leibler-divergence*

$$\mathbb{KL}\left(\{g_{ji}^{\text{obs}}\}_i \,;\, I_0 \exp(-\mathscr{P}_j(f))\right) \approx \int_{\bigcup_i D_i} \left(I_0 \exp(-\mathscr{P}_j(f)) - \left(\textstyle\sum_i g_{ji} \mathbf{1}_{D_i}\right) \cdot \left(1 - \mathscr{P}_j(f)\right)\right) dx$$

[5] T. Hohage and F. Werner (2016). *Inverse Problems* **32**(9):093001.

Computed Tomography with Poisson Noise

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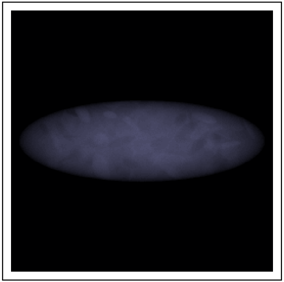

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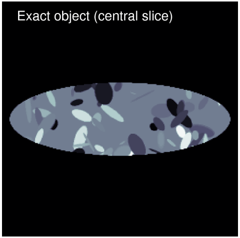
$$=: \mathcal{S}_j^{\mathrm{expKL}}\left(g_j^{\mathrm{obs}};\, \mathscr{P}_j(f)\right) \rightsquigarrow \text{apply in SART-scheme}$$

[5] T. Hohage and F. Werner (2016). *Inverse Problems* **32**(9):093001.

Poisson Noise-Sensitive SART

Simulation:

- Illumination I_0: 10^4 photons per pixel
- Ellipsoidal object f: mean absorption along short half-axes: 90 %, long half-axis: 99.9 %
- 3D circular cone-beam data



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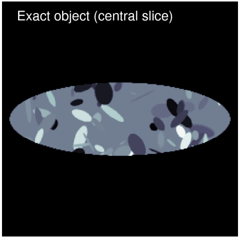
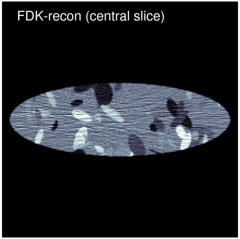


Feldkamp-Davis-Kress:

$$p_j^{obs} = -\ln(g_j^{obs}/I_0)$$

$$f^{rec} = \mathrm{FDK}(p^{obs})$$



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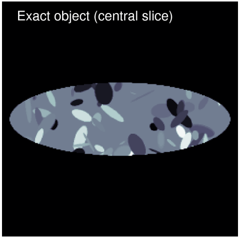

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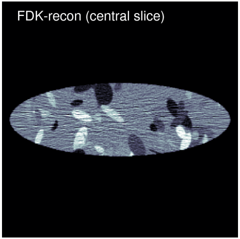
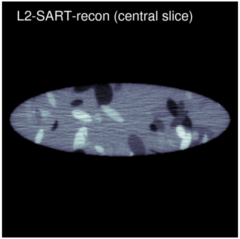


SART with L^2-data fit:

$$f_{k+1} = \mathrm{argmin}_f \left\{ \left\| \mathscr{P}_{j_k}(f) - p_{j_k}^{obs} \right\|_{L^2}^2 + \beta \|f - f_k\|_{L^2}^2 + \alpha \|f\|_{L^2}^2 \right\} \quad \text{(3 cycles)}$$



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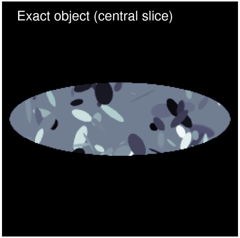

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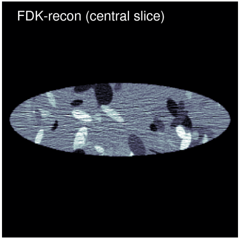

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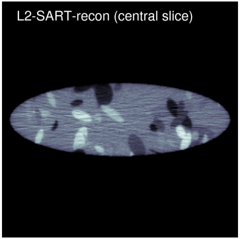
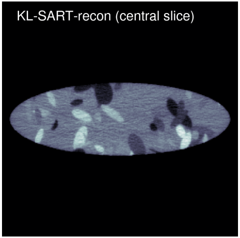


SART with KL-data fit:

$$f_{k+1} = \mathrm{argmin}_f \left\{ \mathcal{S}_{j_k}^{expKL}\left(g_{j_k}^{obs}; \mathscr{P}_{j_k}(f)\right) + \beta \|f - f_k\|_{L^2}^2 + \alpha \|f\|_{L^2}^2 \right\} \quad \text{(3 cycles)}$$



Conclusions

Summary and Conclusions

Kaczmarz-iterations by generalized SART-scheme:

$$f_{k+1} = \operatorname{argmin}_{f\in X} \mathcal{S}_{j_k}(g^{\text{obs}}_{j_k}; F_{j_k}(\mathscr{P}_{j_k}(f))) + \mathcal{R}_k(f) \qquad \text{(genKaczmarz)}$$

1. Forward-project $\rightsquigarrow \mathcal{O}(N^3)$: $p_k = \mathscr{P}_{j_k}(f_k)$
2. Optimize *on projection space* $\rightsquigarrow \mathcal{O}(N^2 \dots N^3)$:
   $$\Delta p_k \in \operatorname{argmin}_{p\in Y} \left\{\mathcal{S}_{j_k}\left(g^{\text{obs}}_{j_k}; F_{j_k}(\mathscr{P}(f^{\text{ref}}_k) + p)\right) + \mathcal{R}_k(f^{\text{ref}}_k + \mathscr{P}^*(p))\right\}$$
3. Back-project increment $\rightsquigarrow \mathcal{O}(N^3)$: $f_{k+1} = f_k + \mathscr{P}^*_{j_k}(\Delta p_k)$

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  $\rightsquigarrow$ other regularization by alternating proximal steps [2] or surrogate functionals? [4]

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[2] M. S. Andersen and P. C. Hansen (2014). *Numerical Algorithms* **67**(1):121-144.

[4] M. Defrise, C. Vanhove and X. Liu (2011). *Inverse Problems* **27**(6):065002.

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Generalized SART-Theorem

Theorem 1 (Generalized SART)

Let $\mathscr{P} : X \to Y$ be linear-bounded on Hilbert spaces X, Y s.t. $\mathscr{P}^*$ has closed range. Let $\mathcal{R} : X \to \bar{\mathbb{R}}$, $\tilde{\mathcal{S}} : Y \to \bar{\mathbb{R}}$ be functionals. Let

$$f^\dagger \in \operatorname*{argmin}_{f \in X} \left(\tilde{\mathcal{S}}\left(\mathscr{P}(f)\right) + \mathcal{R}(f) \right).$$

Moreover, for some $f^{\mathrm{ref}} \in X$, let $\mathcal{R}$ satisfy for all $p \in Y$, $f_0 \in kern(\mathscr{P})$

$$\mathcal{R}(f^{\mathrm{ref}} + \mathscr{P}^*(p) + f_0) \geq \mathcal{R}(f^{\mathrm{ref}} + \mathscr{P}^*(p)), \tag{A1}$$

Then there exists a (possibly distinct) global minimizer $\tilde{f}^\dagger$ given by

$$p^{\mathrm{ref}} = \mathscr{P}(f^{\mathrm{ref}}) \tag{genSART-1}$$

$$\Delta p \in \operatorname*{argmin}_{p \in Y} \left(\tilde{\mathcal{S}}\left(p^{\mathrm{ref}} + \mathscr{P}\mathscr{P}^*(p)\right) + \mathcal{R}(f^{\mathrm{ref}} + \mathscr{P}^*(p)) \right) \tag{genSART-2}$$

$$\tilde{f}^\dagger = f^{\mathrm{ref}} + \mathscr{P}^*(\Delta p) \tag{genSART-3}$$

Proof of Generalized SART-Theorem

Proof.

- Since $\mathscr{P}^*(Y)$ is closed, we may decompose $f^\dagger - f^{\text{ref}} = \mathscr{P}^*(\Delta p) + f_0$ for some $\Delta p \in Y$, $f_0 \in \text{kern}(\mathscr{P})$
- Setting $\tilde{f}^\dagger := f^{\text{ref}} + \mathscr{P}^*(\Delta p)$, we have by assumption

$$\mathcal{R}(f^\dagger) = \mathcal{R}(f^{\text{ref}} + \mathscr{P}^*(\Delta p) + f_0) \geq \mathcal{R}(f^{\text{ref}} + \mathscr{P}^*(\Delta p)) = \mathcal{R}(\tilde{f}^\dagger)$$

$$\tilde{\mathcal{S}}\left(\mathscr{P}(f^\dagger)\right) = \tilde{\mathcal{S}}\left(\mathscr{P}(\tilde{f}^\dagger + f_0)\right) = \tilde{\mathcal{S}}\left(\mathscr{P}(\tilde{f}^\dagger)\right)$$

- Thus, $\tilde{\mathcal{S}}\left(\mathscr{P}(f^\dagger)\right) + \mathcal{R}(f^\dagger) \geq \tilde{\mathcal{S}}\left(\mathscr{P}(\tilde{f}^\dagger)\right) + \mathcal{R}(\tilde{f}^\dagger)$
- Since $\tilde{f}^\dagger = f^{\text{ref}} + \mathscr{P}^*(\Delta p)$ is a minimizer, we obtain

$$\mathscr{P}^*(\Delta p) \in \underset{\Delta f \in X}{\text{argmin}}\left(\tilde{\mathcal{S}}\left(\mathscr{P}(f^{\text{ref}}) + \mathscr{P}(\Delta f)\right) + \mathcal{R}(f^{\text{ref}} + \Delta f)\right)$$

$$\Rightarrow \quad \Delta p \in \underset{p \in Y}{\text{argmin}}\left(\tilde{\mathcal{S}}\left(\mathscr{P}(f^{\text{ref}}) + \mathscr{P}\mathscr{P}^*(p)\right) + \mathcal{R}(f^{\text{ref}} + \mathscr{P}^*(p))\right) \qquad \square$$

Convergence Theory by Kindermann and Leitao I

Theorem 2 (Kindermann & Leitao (2014) [6])

Let $X, Y_1, \dots, Y_M$ Hilbert spaces, $A_j : X \to Y_j$ linear-bounded, $f_0 \in X$ and $g^{\text{obs}} = (g^{\text{obs}}, \dots, g^{\text{obs}}_M)^T$ observations Set $Y = \prod_j Y_j$ and $A := (A_1, \dots, A_M)$. Then, after performing a symmetric Kaczmarz-cycle

$$f_k = \begin{cases} \operatorname{argmin}_{f \in X} \|A_k(f) - g_k^{\text{obs}}\|^2_{Y_j} + \beta \|f - f_{k-1}\|^2_X & \text{for } k \le M \\ \operatorname{argmin}_{f \in X} \|A_{2M-(k-1)}(f) - g^{\text{obs}}_{2M-(k-1)}\|^2_{Y_j} + \beta \|f - f_{k-1}\|^2_X & \text{for } k \ge M+1 \end{cases}$$

the final iterate minimizes the Tikhonov-functional

$$f_{2M} = \operatorname*{argmin}_f \|M_{\text{SB}} \cdot (A(f) - g^{\text{obs}})\|^2_Y + \frac{\beta}{2}\|f - f_0\|^2_X$$

$$M_{\text{SB}} = \begin{pmatrix} \text{id} + \frac{1}{2\beta} A_1 A_1^* & & 0 \\ & \ddots & \\ 0 & & \text{id} + \frac{1}{2\beta} A_M A_M^* \end{pmatrix}^{1/2} \begin{pmatrix} \text{id} & -\frac{1}{\beta} A_1 A_2^* & \dots & -\frac{1}{\beta} A_1 A_M^* \\ & \ddots & \ddots & \vdots \\ 0 & & \ddots & -\frac{1}{\beta} A_{M-1} A_M^* \\ & & & \text{id} \end{pmatrix}^{-1}$$

[5] S. Kindermann and A. Leitao (2014). *Inverse Problems and Imaging* **8**(1):149–172

Convergence Theory by Kindermann and Leitao II

Theorem 2 (Kindermann & Leitao (2014) [6])

[...] the final iterate minimizes the Tikhonov-functional

$$f_{2M} = \operatorname*{argmin}_f \|M_{\text{SB}} \cdot (A(f) - g^{\text{obs}})\|_Y^2 + \frac{\beta}{2}\|f - f_0\|_X^2$$

$$M_{\text{SB}} = \begin{pmatrix} \text{id}+\frac{1}{2\beta}A_1A_1^* & & 0 \\ & \ddots & \\ 0 & & \text{id}+\frac{1}{2\beta}A_MA_M^* \end{pmatrix}^{1/2} \begin{pmatrix} \text{id} & -\frac{1}{\beta}A_1A_2^* & \dots & -\frac{1}{\beta}A_1A_M^* \\ & \ddots & \ddots & \vdots \\ 0 & & \ddots & -\frac{1}{\beta}A_{M-1}A_M^* \\ & & & \text{id} \end{pmatrix}^{-1}$$

Interpretation:

- Symmetric Kaczmarz minimizes the corresponding full Tikhonov functional except for modified Y-space metric ($\rightsquigarrow$ *preconditoned*)
- Deviation from true Tikhonov minimizer is $\mathcal{O}(\beta^{-1})$
- Result depends on ordering of the operators $A_1, \dots, A_M$
- Identification with Tikhonov-reg. enables convergence theory [6]

[5] S. Kindermann and A. Leitao (2014). *Inverse Problems and Imaging* **8**(1):149–172

A Posteriori Estimates: Convex Theory

Setting: convex data-term + global quadratic penalty

$\tilde{\mathcal{S}}_k(f) := \frac{1}{\beta}\mathcal{S}_{j_k}\left(g^{\text{obs}}_{j_k}; F_{j_k}(\mathscr{P}_{j_k}(f))\right) + \frac{\alpha}{\beta M}\|f\|^2$ convex, proper, lower-semicont.

$$f_{k+1} = \operatorname*{argmin}_{f\in X} \mathcal{S}_k(g^{\text{obs}}_{j_k}; F_{j_k}(\mathscr{P}_{j_k}(f))) + \beta\|f - f_k\|^2 + \frac{\alpha}{M}\|f\|^2$$

$$= \operatorname*{argmin}_{f\in X} \tilde{\mathcal{S}}_{k \bmod M+1}(f) + \|f - f_k\|^2 = \left(\mathrm{id} + \partial\tilde{\mathcal{S}}_{k \bmod M+1}\right)^{-1}(f_k) =: T_k(f_k)$$

- Proximal operators are *contractive*: $\|T_k(f) - T_k(g)\| \le \frac{\alpha}{\beta M}\|f - g\|$
- Convergence to *limit cycle*: $f^\dagger_{k+M} = f^\dagger_k$ for all $k = 1, \dots, M$
  with linear rate: $\|f_{k+M} - f^\dagger_k\| \le \exp(-\alpha/\beta)\|f_k - f^\dagger_k\|$
- Optimality condition for cycle: $0 = \sum_{k=1}^M f^\dagger_{k-1} - f^\dagger_k \in \sum_{k=1}^M \partial\tilde{\mathcal{S}}_k(f^\dagger_k)$

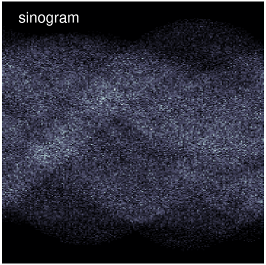
Convergence analysis approach:

Cycle $\{f^\dagger_k\}$ scatters around bulk Tikhonov minimizer $f^\dagger : 0 \in \sum_{k=1}^M \partial\tilde{\mathcal{S}}_k(f^\dagger)$.

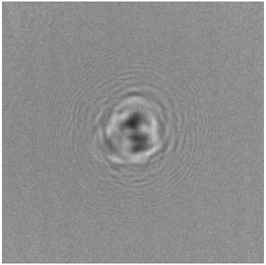
Further Applications

1. Kullback-Leibler-SART for few-photon data

$$f_{k+1} = \mathrm{KL}\left(g_{j_k}^{\mathrm{obs}};\ \mathscr{P}_{j_k}(f)\right) + \beta\|f - f_k\|_{H^1}^2 + \frac{\alpha}{M}\|f\|_{H^1}^2$$



Further Applications



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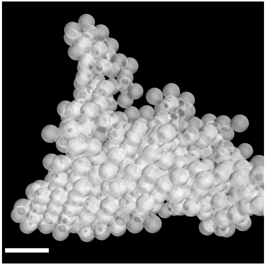

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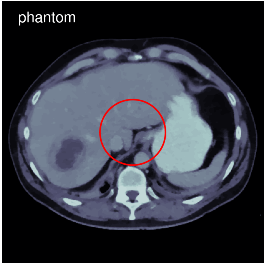

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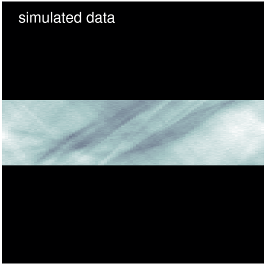

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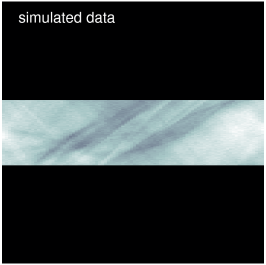

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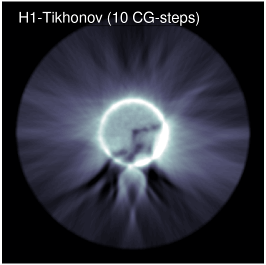

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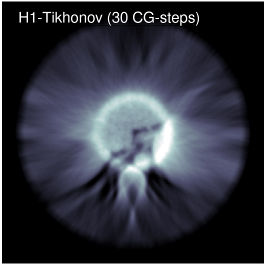

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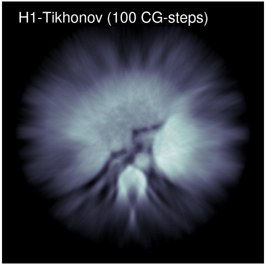

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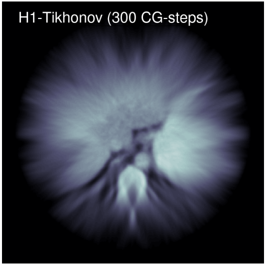

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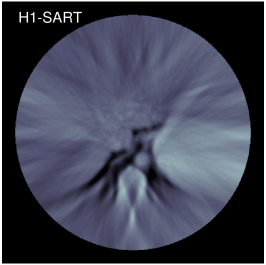

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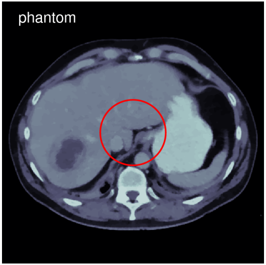

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