Phase Retrieval in X-Ray Propagation Imaging - An Unstable Technique?

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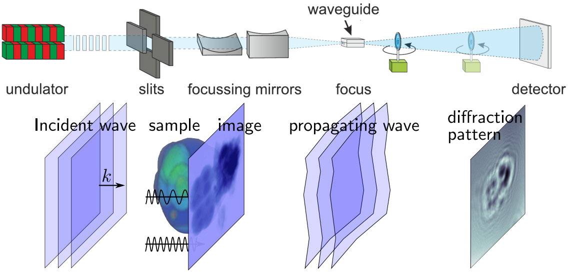
Imaging Setup: X-Ray Phase Contrast



Forward model: $$F : h = \underbrace{-\mathrm{i}\phi - \mu}_{\substack{\textbf{Phase shifts} \\ \text{(+absorption)}}} \mapsto I = \Big| \underbrace{\mathcal{D}}_{\substack{\text{Fresnel} \\ \text{propagator}}} \underbrace{(\exp(-\mathrm{i}\phi - \mu))}_{\text{exit wave field}} \Big|^2$$

Phase Reconstruction:

Recover the object h from noisy data $I^{\varepsilon} = F(h) + \varepsilon$ (+ *a priori* knowledge)

- Is h uniquely determined by $F(h)$? Yes, if h has compact support![1]
- Robust to measurement errors? *This talk!* ☺

[1] SCM (2015). Inverse Problems **31** 065003

The Stability Problem

Stability: Changes of the image h result in finite perturbations of the intensity data $\rightsquigarrow$ *objects can be distinguished*

$$\|F(h_1) - F(h_2)\|_{(L^2)} \geq C\|h_1 - h_2\|_{(L^2)} \quad \text{for all} \quad h_1, h_2 \in A$$

- Reconstruction error $\leq C^{-1}\|\varepsilon\| \Rightarrow$ *stable recovery* (even *well-posed*)
- Achievable quality is determined by the constant $C > 0$
- What is a suitable set of admissible objects A? ($\rightsquigarrow$ *constraints*)

Linear Weak Object Limit:

Restrict to weakly phase shifting / absorbing objects $\phi, \mu \ll 1$. Then

$$F(h_1) - F(h_2) \approx \mathcal{D}(h_1 - h_2) + \mathcal{D}^{-1}(\overline{h_1 - h_2}) =: T\Delta h \qquad \text{(linear)}$$

- C = smallest singular value of $T \rightsquigarrow$ *simpler computation + analysis*
- Weak object linearization widely used in reconstructions ($\rightsquigarrow$ CTF)

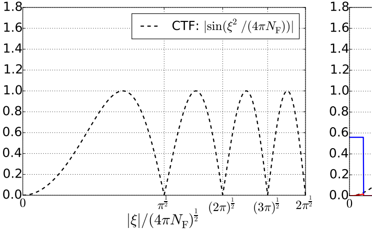
Pure Phase Objects I

Constraint: Thin light-element samples are transparent to hard X-rays

$\Rightarrow$ restrict to *pure phase objects*: $h = -\mathrm{i}\phi - \not{\mu}$

$$Th = \mathcal{D}h + \mathcal{D}^{-1}\overline{h} = 2\mathcal{F}^{-1}\Big(\underbrace{\sin\Big(\frac{\xi^2}{4\pi N_\mathrm{F}}\Big)}_{=s(\xi)} \cdot \mathcal{F}(\phi)\Big) =: S\phi \qquad (\mathcal{F}\text{: Fourier transform})$$

- Object Fourier components multiplied by *contrast transfer function s* (CTF)
- Fresnel number N_F governs diffraction ($N_\mathrm{F} = \infty$: ray optics, $N_\mathrm{F} = 0$: far-field)
- No contrast in Fourier-frequencies ξ that are CTF-zeros $\rightsquigarrow$ *stability?*



General Instability:

For objects $A = L^2(\mathbb{R}^2)$, the inversion of S is unique but not stable:

$$S\phi = 0 \Leftrightarrow \phi = 0 \quad \text{but} \quad \inf_{\phi \in A, ||\phi||=1} ||S\phi|| = 0$$

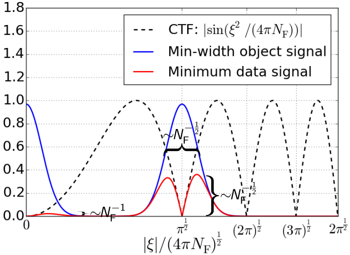
Pure Phase Objects II

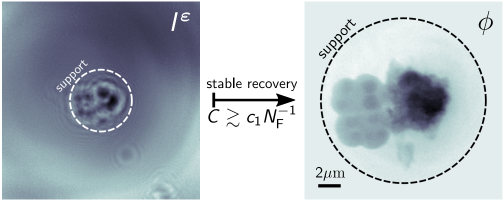
Observation:

Image reconstruction is unstable to perturbations peaked at CTF-zeros.

- *Support constraint:* object is contained in a known subdomain Ω of the image
- *Uncertainty principle:* Confinement in real-space imposes *minimum* width in Fourier space: $\sigma_{|\mathcal{F}(h)|^2} \gtrsim 1/\sigma_{|h|^2} \geq \frac{2}{\mathrm{diam}(\Omega)}$

$\Rightarrow$ *Stable*: $C_{\mathrm{phase}} := \inf_{\substack{\phi \in L^2(\Omega) \\ \|\phi\|=1}} \|S\phi\| \gtrsim c_1 N_F^{-1}$





Theorem 1:

Let N_F be the Fresnel number w.r.t. $\mathrm{diam}(\Omega)$. Then the recovery of *pure phase objects* $\phi \in L^2(\Omega)$ is stable with

$$C_{\mathrm{phase}} \geq \min\{C_1, c_1 N_F^{-1}\}$$

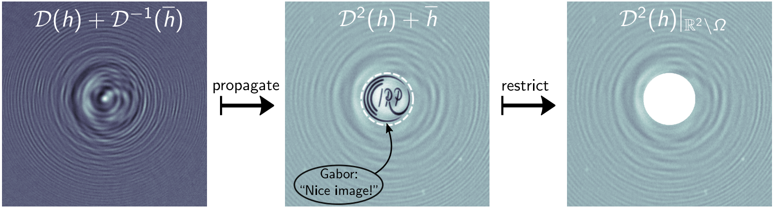
General Objects I

General Objects: induce phase shifts ϕ *and* absorption μ

$$Th = \underbrace{\mathcal{D}(h)}_{\text{image}} + \underbrace{\mathcal{D}^{-1}(\overline{h})}_{\text{twin image}} = 2\mathcal{F}^{-1}\left(\sin\left(\frac{\xi^2}{4\pi N_F}\right)\cdot\mathcal{F}(\phi)\right) - 2\mathcal{F}^{-1}\left(\cos\left(\frac{\xi^2}{4\pi N_F}\right)\cdot\mathcal{F}(\mu)\right)$$

- Sum of CTF-contributions in ϕ and μ $\rightsquigarrow$ *how to separate?*
- *Alternative point of view:* Superposition of propagated image and back-propagated twin image of h $\rightsquigarrow$ *indistinguishable?*

Idea: Inverse Gabor Holography



Question: (How well) can we reconstruct the missing data $\mathcal{D}^2(h)|_\Omega$?

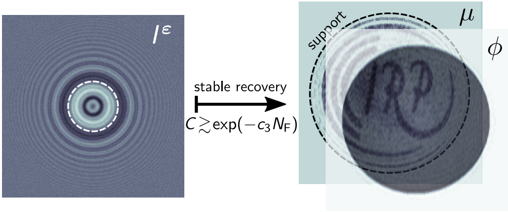
General Objects II

Stability Analysis:

$$\|Th\| = \|\mathcal{D}Th\| = \|\mathcal{D}^2(h) + \overline{h}\| \geq \|\mathcal{D}^2(h)|_{\mathbb{R}^m\setminus\Omega}\| = \|\mathcal{F}(m_F \cdot h)|_{\mathbb{R}^2\setminus(\pi N_F \Omega)}\|$$

$$\Rightarrow C_{\text{gen}} := \inf_{\substack{h\in L^2(\Omega)\\ \|h\|=1}} \|Th\| \geq \inf_{\substack{h\in L^2(\Omega)\\ \|h\|=1}} \|\mathcal{F}(h)|_{\mathbb{R}^2\setminus(\pi N_F \Omega)}\|$$

- Inverting T at least as stable as $\mathcal{F}(h)|_{\mathbb{R}^2\setminus(\pi N_F\Omega)} \mapsto h$ (*FT completion*)
- Implies uniqueness: if $h \neq 0$ is compactly supported, $\mathcal{F}(h)$ is not
- Smallest singular value of incomplete FT is $\gtrsim \exp(-cN_F) \rightsquigarrow$ *stable*[1]



Theorem 2:

Let N_F be the Fresnel number w.r.t. diam(Ω). Then the recovery of *general objects* $h \in L^2(\Omega)$ is stable with

$$C_{\text{gen}} \geq C_3 \exp(-c_3 N_F)$$

[1] SCM, Bartels M, Krenkel M, Salditt T, and Hohage T. (2016). *Optics Express* (accepted).

Conclusions

Summary and Implications:

- ✓ Stability results for X-ray propagation imaging $\rightsquigarrow$ robust to noise
- ✓ Ingredients: support constraint + uncertainty principles
- ✓ *Pure phase objects* $C_{\mathrm{phase}} \sim N_{\mathrm{F}}^{-1}$: stable for deeply holographic data (no regularization!), reconstruction of ϕ feasible also for larger N_{F}
- ✓ *General objects* $C_{\mathrm{gen}} \sim \exp(-cN_{\mathrm{F}})$: recovery of phase + *absorption* ϕ, μ from *single* image is feasible *if and only if* $N_{\mathrm{F}} \lesssim 10$

Future Work:

- Numerical Prediction of stability constants C_* and least stable modes for given Fresnel number N_{F}
- Stability analysis for the full *nonlinear* imaging model
- Stable reconstruction methods with and without support constraints