# Characterizations of Variational Source Conditions, Converse Results and Maxisets of Spectral Regularization Methods

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joint work with Thorsten Hohage

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# Outline

1. Variational Source Conditions
2. Convergence Rates in Hilbert Spaces
3. Banach spaces: Examples

# Outline

# Classical Setup

Setup:

- Let $\mathbb{X}, \mathbb{Y}$ be Banach spaces
- $T : \mathbb{X} \to \mathbb{Y}$ be a (non-)linear operator
- $f^\dagger \in \mathbb{X}$ the true solution
- noisy measurement $g^{\mathrm{obs}}$

$$g^{\mathrm{obs}} = Tf^\dagger + \xi, \qquad \|\xi\| \leq \delta$$

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Problems:

- find approximation $f_\alpha^\delta$ of $f^\dagger$ from $g^{\mathrm{obs}}$ $\rightsquigarrow$ regularization methods
- estimate distance of $f_\alpha^\delta$ to $f^\dagger$

# Variational Source Conditions

Tikhonov Regularization:

$$f_\alpha^\delta \in \arg\min \left[ \frac{1}{2\alpha} \left\| Tf - g^\delta \right\|_{\mathbb{Y}}^2 + \frac{1}{r} \left\| f \right\|_{\mathbb{X}}^r \right]$$

Variational source conditions (VSC) for a concave index function $\psi$:

$$\forall f : \quad \left\langle f^*, f^\dagger - f \right\rangle_{\mathbb{X}} \leq \Delta_{\frac{1}{r}\|\cdot\|_{\mathbb{X}}^r}(f, f^\dagger) + \psi\left( \left\| F(f) - F(f^\dagger) \right\|_{\mathbb{Y}}^2 \right)$$

with $f^* \in \partial \frac{1}{r} \left\| f^\dagger \right\|_{\mathbb{X}}^r$ implies rate of convergence

$$\Delta_{\frac{1}{r}\|\cdot\|_{\mathbb{X}}^r}(f_\alpha^\delta, f^\dagger) \leq \psi\left(\delta^2\right)$$

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- Let $P_k\colon \mathbb{X} \to \mathbb{X}$ be a family of projections and quantify
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$$\left\langle f^*, P_k(f^\dagger - f)\right\rangle \leq \sigma(k)\left\|Tf^\dagger - Tf\right\|_{\mathbb{Y}} + \gamma\kappa(k)\left\|f^\dagger - f\right\|_{\mathbb{X}}$$

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Then $f^\dagger$ fulfills a VSC with index function

$$\psi_{\mathrm{vsc}}(t) = \inf_{k \in K}\left[\sigma(k)\sqrt{t} + \frac{1}{r'}\left(\frac{2}{C_\Delta}\right)^{\frac{r'}{r}}(1+\gamma)^{r'}\kappa(k)^{r'}\right].$$

T. Hohage and F. Weidling. *Characterizations of Variational Source Conditions, Converse Results, and Maxisets of Spectral Regularization Methods.* **SIAM JNA**, 55:598–620, 2017.

# Example: Compact Operators

Let $\mathbb{X}, \mathbb{Y}$ be Hilbert spaces, $T$ compact with singular system $(f_j, g_j, \sigma_j)_{j\in\mathbb{N}}$

$$P_k f = \sum_{j\leq k} \langle f, f_j \rangle f_j \qquad Q_k g = \sum_{j\leq k} \langle g, g_j \rangle g_j$$

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# Outline

# Assumptions on spectral regularization

$$f_\alpha^\delta := R_\alpha g^{\mathrm{obs}} \qquad \text{with} \qquad R_\alpha = q_\alpha(T^*T)T^*$$

## Assumptions on SR

With $r_\alpha(\lambda) := 1 - \lambda q_\alpha(\lambda)$ assume that for all $\lambda \in \sigma(T^*T)$ and $0 < \alpha \leq \overline{\alpha}$

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1. $|q_\alpha(\lambda)| \leq \frac{C_1}{\alpha}$ for some $C_1 > 0$,
2. $\lambda \mapsto r_\alpha(\lambda)$ is decreasing and $r_\alpha(\lambda) \geq 0$,
3. $\lim_{\alpha \to 0} r_\alpha(\lambda) = 0$ and

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4. $\alpha \mapsto r_\alpha(\lambda)$ is increasing,
5. $0 < C_2 \leq \sup_{0 < \alpha \leq \overline{\alpha}} r_\alpha(\alpha) \leq C_3 < 1$.

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| include | exclude |
|---|---|
| $k$-iterated Tikhonov | spectral cut-off |
| Landweber | $\nu$-methods |
| Showalter | |
| Lardy | |
| modified spectral cut-off | |

# Overview

general strategy

assumptions on regularization methods

Projection approach

$f^\dagger$ VSC with $\psi_\kappa$

$\kappa \in o(\sqrt{t})$

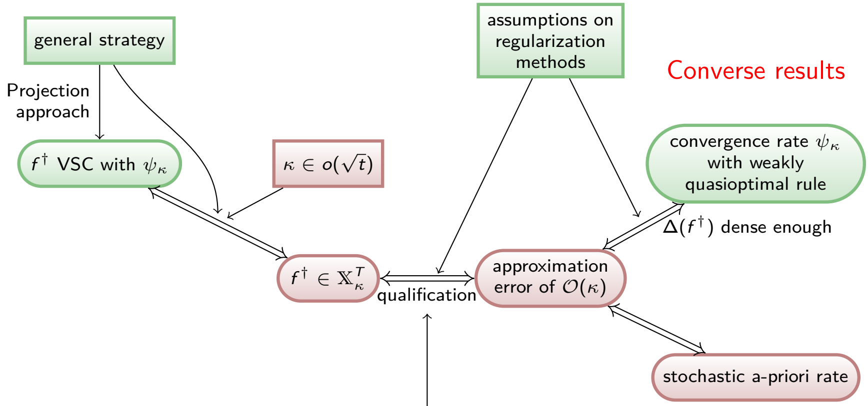
$\psi_\kappa(t) = \kappa(\Theta_\kappa^{-1}(\sqrt{t}))$
$\Theta_\kappa(t) = \sqrt{\lambda}\kappa(\lambda)$

$f^\dagger \in \mathbb{X}_\kappa^T$

$\Longleftrightarrow$

$\sup_{\lambda>0} \frac{1}{\kappa(\lambda)} \|1_{[0,\lambda)}(T^*T)f^\dagger\|_{\mathbb{X}} < \infty$

# Overview




V. Albani, P. Elbau, M.V. de Hoop, O. Scherzer.
*Optimal convergence rates results for linear inverse problems in Hilbert spaces.*
**Numerical Functional Analysis and Optimization** 37:521–540, 2016.

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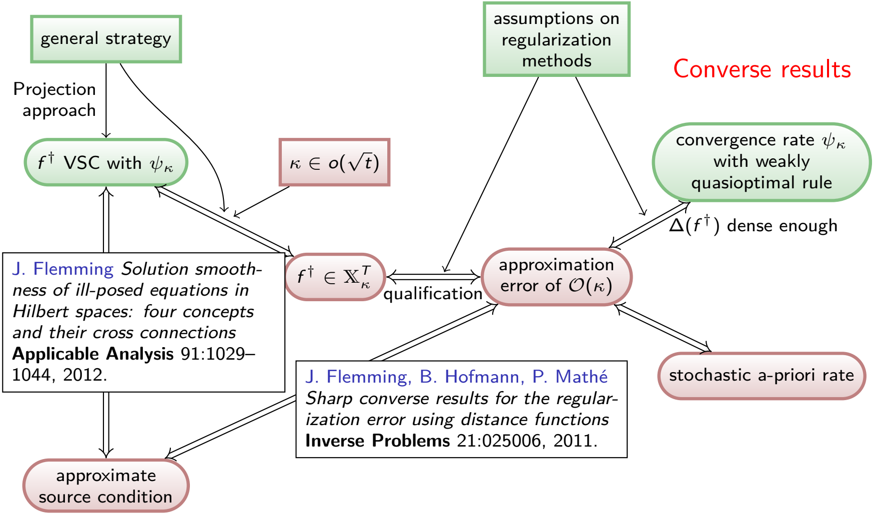
general strategy
Projection approach
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Converse results
$f^\dagger$ VSC with $\psi_\kappa$
$\kappa \in o(\sqrt{t})$
convergence rate $\psi_\kappa$ with weakly quasioptimal rule
$\Delta(f^\dagger)$ dense enough
$f^\dagger \in \mathbb{X}_\kappa^T$
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J. Flemming *Solution smoothness of ill-posed equations in Hilbert spaces: four concepts and their cross connections* **Applicable Analysis** 91:1029–1044, 2012.
J. Flemming, B. Hofmann, P. Mathé *Sharp converse results for the regularization error using distance functions* **Inverse Problems** 21:025006, 2011.
stochastic a-priori rate
approximate source condition

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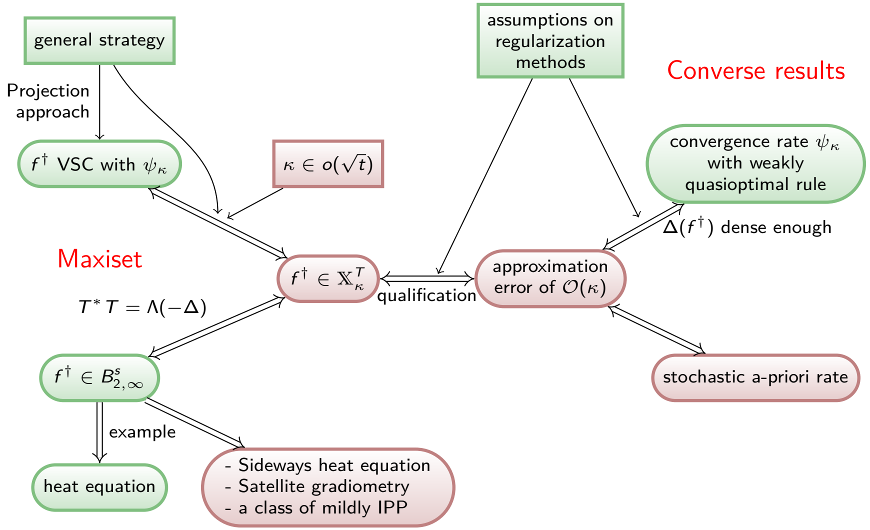
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stochastic a-priori rate
example
heat equation
- Sideways heat equation
- Satellite gradiometry
- a class of mildly IPP

# Parameter choice rules

A parameter choice rule $\alpha_*$ is called

- weakly quasioptimal

- strongly quasioptimal

T. Raus, U. Hämarik. *On the quasioptimal regularization parameter choices for solving ill-posed problems.* **J. Inv. Ill-Posed Probl.** 15:419–439, 2007.

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Examples:

- discrepancy principle: strongly quasioptimal for methods with infinite qualification
- Lepskiĭ: weakly quasioptimal

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# Interchangeability result

**Lemma**

*For all* $\delta \in \Delta(f^\dagger) := \{\|r_\alpha(T^*T)f^\dagger\|/\|R_\alpha\| : 0 < \alpha < \overline{\alpha}\}$ *we have*

$$\inf_{0<\alpha<\overline{\alpha}} \sup_{\|\xi\|\leq\delta} \left\|R_\alpha(Tf^\dagger+\xi)-f^\dagger\right\| \leq 2\sqrt{2} \sup_{\|\xi\|\leq\delta} \inf_{0<\alpha\leq\overline{\alpha}} \left\|R_\alpha(Tf^\dagger+\xi)-f^\dagger\right\|$$

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Corollary:

- In many cases weak and strong quasioptimality coincide.

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## Theorem

*Let* $\kappa(r\alpha) \leq r^p\kappa(\alpha)$ *for some* $p \geq 1$ *and all* $r \geq 1$. *Then for any finite* $\delta_0 > 0$ *the following is equivalent for all considered methods, all* $f^\dagger \neq 0$, *and all weakly quasioptimal parameter choice rules* $\alpha_*$*:*

1. $\sup_{0<\alpha\leq\overline{\alpha}} \frac{1}{\kappa(\alpha)^2}\|r_\alpha(T^*T)f^\dagger\|^2 < \infty$.
2. $\sup_{0<\delta\leq\delta_0} \frac{1}{\psi_\kappa(\delta^2)} \sup_{\|\xi\|\leq\delta} \left\|R_{\alpha_*}(Tf^\dagger + \xi) - f^\dagger\right\|^2 < \infty$.

# Besov Spaces

Besov space $B^s_{p,q}$ with $p \in (1,\infty), q \in [1,\infty], s \in \mathbb{R}$:

$$f_j(x) := \mathcal{F}^*\left(\chi_j \mathcal{F} f\right)(x), \qquad \chi_j(x) := \chi_0(2^{-j}x) - \chi_0(2^{-j+1}x)$$

with $\chi_0(x)$ the characteristic function of the unit ball in $\mathbb{R}^d$

$$\|f\|^q_{B^s_{p,q}} := \sum_{j \in \mathbb{N}_0} 2^{jsq} \|f_j\|^q_{L^p}$$

H. Triebel. *Theory of function spaces I*, Springer, 2010.

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- Relation to other Spaces:

$$B^0_{p,\min\{p,2\}} \subset L^p \subset B^0_{p,\max\{p,2\}}, \qquad B^s_{p,p} = W^{s,p}, \text{ if } s \notin \mathbb{Z}$$

- for $s \in \mathbb{R}$, $\varepsilon > 0$ and $1 \le r \le q \le \infty$

$$B^{s+\varepsilon}_{p,q} \subset B^{s+\varepsilon}_{p,\infty} \subset B^s_{p,1} \subset B^s_{p,r} \subset B^s_{p,q}$$

# Maxisets

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**Theorem**

*Let $\Delta$ be a Laplace-Beltrami operator on $\Omega$ ("sufficiently nice"), $\Lambda : [0, \infty) \to (0, \infty)$ continuous and monotonically decreasing with $\lim_{\mu\to\infty} \Lambda(\mu) = 0$. Let $T : \mathbb{X} := L^2(\Omega) \to \mathbb{Y}$ be bounded such that*

$$T^*T = \Lambda(-\Delta) \quad \text{and set} \quad \kappa(\alpha) = (\Lambda^{-1}(\alpha))^{-1/2}$$

*Then $\mathbb{X}_{\kappa^s}^T = B_{2,\infty}^s(\Omega)$ for all $s > 0$ with equivalent norms.*

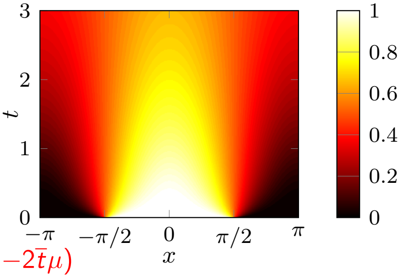
Proof based on:

R. Andreev. *Tikhonov and Landweber convergence rates: characterization by interpolation spaces.* **Inverse Problems** 31:105007, 2015.

# Backward heat equation

$$\partial_t u = \Delta u \qquad \text{in } \mathbb{S}^1 \times (0, \overline{t})$$
$$u(\cdot, 0) = f \qquad \text{on } \mathbb{S}^1$$

unknown: initial temperature $f$
observations: $g = u(\cdot, \overline{t})$, final temperature

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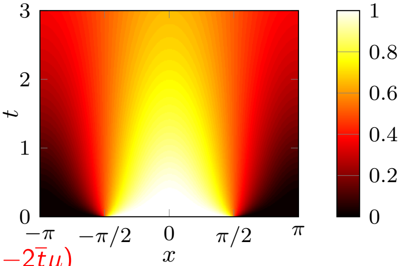


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## Theorem

*The following statements are equivalent for $s > 0$ and $f^\dagger \neq 0$:*

1. $f^\dagger \in B^{2s}_{2,\infty}(\mathbb{S}^1)$
2. *$f^\dagger$ satisfies a VSC with $\psi(t) = C \log(3 + t^{-1})^{-2s}$, $C > 0$.*
3. *For a weakly quasioptimal parameter choice rule $\alpha_*$ we have* $\sup\{\|R_{\alpha_*} g^{\text{obs}} - f^\dagger\| : \|g^{\text{obs}} - Tf^\dagger\| \leq \delta\} = \mathcal{O}\left(\log(\delta^{-1})^{-s}\right)$

# Relation to spectral source conditions

Characterization of spectral source conditions known:

$$f^\dagger = \varphi_s(T^*T)w, \quad \varphi_s(\lambda) = (-\ln\lambda)^{-s} \quad \Leftrightarrow \quad f^\dagger \in H^{2s}(\mathbb{S}^1) = B^{2s}_{2,2}(\mathbb{S}^1)$$

T. Hohage. *Regularization of exponentially ill-posed problems.* **Numerical Functional Analysis and Optimization** 21:439–464, 2000.

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For given example:

$$f^\dagger(t) = \begin{cases} 1, & |t| < \frac{\pi}{2}, \\ 0, & \text{else} \end{cases} \quad \Longrightarrow \quad \widehat{f^\dagger}(n) \approx \frac{1}{|n|}$$

$$\Longrightarrow \quad f^* = f^\dagger \in \begin{cases} H^{2s}, & \text{for } s \in [0, 1/4) \Longrightarrow \text{rate of } \mathcal{O}\left(\log(\delta^{-1})^{-s}\right) \\ B^{1/2}_{2,\infty}, & \Longrightarrow \text{rate of } \mathcal{O}\left(\log(\delta^{-1})^{-1/4}\right) \end{cases}$$

# Outline

# Subdifferential Smoothness

Now just Tikhonov regularization:

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with $\mathbb{X} = B^0_{p,q}$ for $1 < p \leq 2$, $1 < q < \infty$.
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Remember: We want

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**Theorem**

*$f^\dagger \in B^s_{p,\infty}$ for $s > 0$ if and only if $f^* \in B^{s(q-1)}_{p',\infty}$.*

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- Leads to convergence rates:

$$\left\| f^\delta_{\alpha,q} - f^\dagger \right\|_{B^0_{p,q}} = \begin{cases} \mathcal{O}\left(\delta^{\frac{s(q-1)}{a+s(q-1)}}\right) & 1 < q \leq 2, 0 < s < \frac{a}{q-1} \\ \mathcal{O}\left(\delta^{\frac{s}{a+s}}\right) & 2 \leq q < \infty, 0 < s < \frac{a}{q-1}. \end{cases}$$

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- For $q \geq 2$ these are the Hilbert rates.
- If in addition $s + a > d\frac{2-p}{2p}$ we can show these are optimal $\rightsquigarrow$ $B^s_{p,\infty}$ are again maxisets.

# Example: Inverse heat equation

- Under the assumption $f^\dagger \in B^{2s}_{p,\infty}$ we obtain via a vsc the rate:

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- Optimality is open
- Note for $f^\dagger$ smooth up to jumps one obtains $f^\dagger \in B^{d/p}_{p,\infty}$
$\rightsquigarrow$ faster convergence then in Hilbert space setting.

# Summary


# Thank you for your attention!

T. Hohage and F. Weidling. *Characterizations of Variational Source Conditions, Converse Results, and Maxisets of Spectral Regularization Methods.* **SIAM JNA**, 55:598–620, 2017