A uniqueness result for propagation-based phase contrast imaging from a single measurement

IFIP TC7.4 Workshop on Inverse Problems and Imaging

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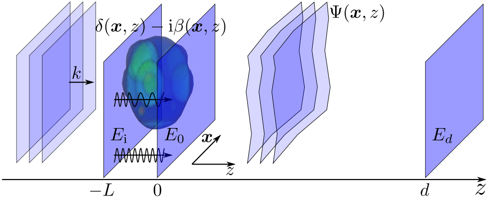
Physical Problem



Monochromatic EM-Waves: $\Delta\Psi + n^2 k^2 \Psi = 0, \; n = 1 - \delta + \mathrm{i}\beta$

- *Interaction*: By geometrical optics $\rightarrow$ *contact image*

$$\Psi(\cdot, 0) = 1 + \left[\exp\left(-\mathrm{i}k \int_{-L}^{0} (\delta - \mathrm{i}\beta) \, \mathrm{d}z\right) - 1\right] = 1 + h$$

- *Fresnel Diffraction*: Propagation of paraxial waves (*no far-field*)

$$\Psi(\cdot, d) = \mathcal{D}_d^{(\mathrm{F})}(\Psi(\cdot, 0)) \propto w^{(\mathrm{F})} \cdot \mathcal{F}\left(w^{(\mathrm{F})} \cdot \Psi(\cdot, 0)\right), \quad w^{(\mathrm{F})} \sim \exp\left(\mathrm{i}\boldsymbol{\xi}^2\right)$$

- *Detected Intensities*: $I = |\Psi(\cdot, d)|^2 = |1 + \mathcal{D}_d^{(\mathrm{F})}(h)|^2$

Inverse Problem

Forward Operator:

$$F : \mathscr{S}_c'(\mathbb{R}^n) \to \mathscr{C}^\infty(\mathbb{R}^n); \; F(h)(\boldsymbol{\xi}) = \left| \exp\left(-\mathrm{i}\boldsymbol{\xi}^2\right) + \mathcal{F}(w^{(\mathrm{F})} \cdot h)(\boldsymbol{\xi}) \right|^2 \quad (1)$$

$\Rightarrow$ Well-defined? Injective?

Inverse Problem: (Phase Retrieval in Phase Contrast Imaging)

Reconstruct the contact image $h^\dagger \in \mathscr{S}_c'(\mathbb{R}^n)$ from intensities $I^\dagger = F(h^\dagger)$.

Previous Results:

- *Contrast transfer function* (linearize F for $h \in L^2(\mathbb{R}^n)$ small):

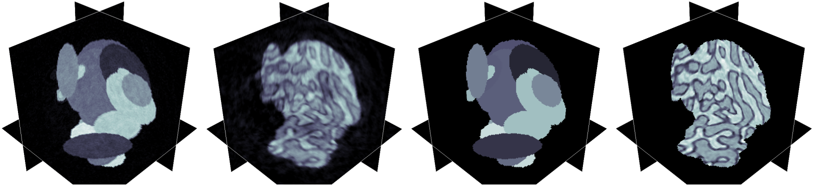
$$\mathcal{F}(F(h) - 1)(\boldsymbol{\xi}) \propto \sin\left(\chi\boldsymbol{\xi}^2\right) \mathcal{F}(\Im(h))(\boldsymbol{\xi}) + \cos\left(\chi\boldsymbol{\xi}^2\right) \mathcal{F}(\Re(h))(\boldsymbol{\xi})$$

- *Complex* h unique from *two* measurements [Jonas and Louis, 2004]
- *Phase vortex* [Nugent, 2007] $\rightarrow$ *single* intensity pattern insufficient?

Numerical Evidence

Phase Contrast Tomography:

- 3D-structure $\delta + \mathrm{i}\beta$ from contact images at different incident angles
- Combined ansatz: Simultaneous phase retrieval and Radon inversion
- Solve $F_{\mathrm{tot}}(\delta + \mathrm{i}\beta) = I$ by iteratively regularized Gauss-Newton



δ: reconstruction β: reconstruction δ: exact β: exact

- Unique reconstruction of compact objects from a single measurement
- Tomographic correlations of contact images facilitate phase retrieval

Paley-Wiener-Schwartz Theorem

Theorem 1 (Paley-Wiener-Schwartz Theorem)

Let $K \subset \mathbb{R}^n$ compact, convex. Then, for any $u \in \mathscr{S}_c'(\mathbb{R}^n)$, $\mathrm{supp}(u) \subset K$, *$\hat{u} := \mathcal{F}(u)$ defines an entire function in $\mathbb{C}^n$ and $\exists\, C > 0, N \in \mathbb{N}_0$ s.t.*

$$|\hat{u}(\boldsymbol{\xi})| \leq C(1 + \|\boldsymbol{\xi}\|_2)^N \exp\left(\sup_{x\in K} \Im(\boldsymbol{\xi}) \cdot x\right) \quad \forall\, \boldsymbol{\xi} \in \mathbb{C}^n \tag{2}$$

Conversely, any entire function $\hat{u}$, satisfying (2), *is the complex extension of the Fourier transform of such a distribution.*

Correspondence:

$$\mathscr{S}_c'(\mathbb{R}^n) \overset{\mathcal{F}}{\longleftrightarrow} \text{entire functions of } \textit{order} \leq 1 \text{ (i.e. } \lesssim \exp(\tau\|\boldsymbol{\xi}\|))$$

- $f := \exp\left(-\mathrm{i}(\cdot)^2\right) + \mathcal{F}(w^{(\mathrm{F})}h)$ entire of order 2 $\Rightarrow$ F well-defined
- $F(h) = f \cdot \overline{f(\bar{\cdot})} =: f \cdot f^*$ entire $\Rightarrow$ uniquely determined by $F(h)_{|U}$
- Uniqueness problem accessible by theory of entire functions!

Entire Functions in 1D

Setting:

- $f : \mathbb{C} \to \mathbb{C}$ entire, not identically zero
- Complex zeros: $Z_f := \{a_j\}_{j \in J} \subset \mathbb{C} \setminus \{0\}, J \subset \mathbb{N}$
- Convergence exponent: $\rho_f := \inf\{\rho \geq 0 : \sum_{j \in J} |a_j|^{-\rho} < \infty\}$
- Rank: $p_f := \min\{p \in \mathbb{N}_0 : \sum_{j \in J} |a_j|^{-(p+1)} < \infty\}$

Theorem 1 (Hadamard's factorization theorem)

Let f be entire of order $\lambda_f < \infty$. Then $p_f \leq \lambda_f$ and

$$f(\xi) = \xi^m \exp(q_f(\xi)) \prod_{j \in J} E_{p_f}\left(\frac{\xi}{a_j}\right) \quad \forall\, \xi \in \mathbb{C} \qquad (3)$$

with $m \in \mathbb{N}_0$, $\deg(q_f) \leq \lambda_f$ and $E_n(z) = (1-z)\exp\left(\sum_{j=1}^{n} \frac{z^j}{j}\right)$.
Conversely, for any sequence Z_f, polynomial q_f and $m \in \mathbb{N}_0$, (3) defines an entire funtion f such that $\lambda_f = \max\{\deg(q_f), \rho_f\}$

Phase Retrieval in 1D

Hadamard Factorization of $|f|^2$

$$|f|^2(\xi) = f \cdot f^*(\xi) = \xi^{2m} \exp(2\Re(q_f)(\xi)) \prod_{j \in J} E_{p_f}\left(\frac{\xi}{a_j}\right) \cdot E_{p_f}\left(\frac{\xi}{\overline{a_j}}\right)$$

- Quantification of the information obtained by measuring $|f|^2$
- Uniqueness theory for (Fourier-)phase retrieval of compact signals [Akutowicz, 1956, Akutowicz, 1957, Walther, 1963]

Lemma 2

Let $f, \tilde{f} : \mathbb{C} \to \mathbb{C}$ entire s.t. $\lambda_{\tilde{f}} \leq \lambda_f < \infty$, $|f|^2_{|U} = |\tilde{f}|^2_{|U}$ for $U \subset \mathbb{R}$ open. Then there exist entire functions $f_1, f_2 : \mathbb{C} \to \mathbb{C}$ of order $\leq \lambda_f$ such that

$$f = f_1 \cdot f_2 \quad \text{and} \quad \tilde{f} = f_1 \cdot f_2^*. \tag{4}$$

Conversely, if f_1 and f_2 are entire of order λ, then f and $\tilde{f}$ are entire functions of order $\leq \lambda$ satisfying $|f|^2_{|\mathbb{R}} = |\tilde{f}|^2_{|\mathbb{R}}$.

Main Result

Theorem 2 (Uniqueness of phase contrast imaging for compact objects)

For $w \in \mathscr{C}^\infty(\mathbb{R}^n)$ everywhere nonzero, $\alpha \in \mathbb{C} \setminus \mathbb{R}$ and $P_0 \in \mathscr{S}'_c(\mathbb{R}^n) \setminus \{0\}$ define

$$F : \mathscr{S}'_c(\mathbb{R}^n) \to \mathscr{C}^\infty(\mathbb{R}^n); \; F(h) = |\mathcal{F}(P_0)\exp(\alpha(\cdot)^2) + \mathcal{F}(w \cdot h)|^2 \quad (5)$$

Then F is well-defined and injective. Moreover, any $h \in \mathscr{S}'_c(\mathbb{R}^n)$ is uniquely determined by $F(h)_{|U}$ on an arbitrary open set $U \subset \mathbb{R}^n$.

General Idea of the Proof:

- ✓ Well-definedness + unique extension $F(h)_{|U} \mapsto F(h)$ by PWS-Thm
- ✓ 1D case: Show that the "factorization-construction" of alternate solutions in Lemma 2 is incompatible with the structure of F
- ✓ Reduce case $n > 1$ to a family of 1D-problems

Proof of the Main Result I

Setting:

- Let $U \subset \mathbb{R}$ open, $h, \tilde{h} \in \mathscr{S}_c'(\mathbb{R})$ s.t. $F(h)_{|U} = F(\tilde{h})_{|U}$
- Define $f, \tilde{f} : \mathbb{C} \to \mathbb{C}$ by $f(\xi) := \mathcal{F}(P_0)(\xi) \exp(\alpha \xi^2) + \mathcal{F}(wh)(\xi)$
- Assume $h \neq \tilde{h}$

- $f, \tilde{f}$ entire of order 2 satisfying $|f|^2_{|U} = F(h)_{|U} = F(\tilde{h})_{|U} = |\tilde{f}|^2_{|U}$
- By Lemma 2: $\exists$ order ≤ 2 entire functions $f_1, f_2 : \mathbb{C} \to \mathbb{C}$ such that

$$f = f_1 \cdot f_2 \quad \text{and} \quad \tilde{f} = f_1 \cdot f_2^*$$

- $g := f_1 \cdot (f_2 - f_2^*) = f - \tilde{f} = \mathcal{F}(w \cdot (h - \tilde{h}))$ is entire of order ≤ 1 and non-zero since $h - \tilde{h} \in \mathscr{S}_c'(\mathbb{R}) \setminus \{0\}$
- g has rank ≤ 1 by Theorem 1

Proof of the Main Result II

- Zeros of f_1 contained in those of $g = f_1 \cdot (f_2 - f_2^*)$
- f_1 of order ≤ 2 and rank $\leq 1 \overset{\text{Hadamard}}{\Rightarrow} \exists\, \mu \in \mathbb{C},\ f_0$ of order ≤ 1
$$f_1 = \exp(\mu(\cdot)^2) \cdot f_0 \qquad (6)$$
- Set $\gamma := \Im(\mu)$, $f_3 := \exp(\Re(\mu)(\cdot)^2) \cdot f_2$ and substitute (6) into g:
$$g \cdot f_0^* = \exp(\mathrm{i}\gamma(\cdot)^2) \cdot (f_0 \cdot f_0^*) \cdot (f_3 - f_3^*) = -\exp(2\mathrm{i}\gamma(\cdot)^2) \cdot g^* \cdot f_0$$
- rhs of order 2, lhs ≤ 1 [Boas, 2011, Ch. 3] $\Rightarrow \gamma = 0$
$$\Rightarrow \quad f = f_0 \cdot f_3 \quad \text{and} \quad \tilde{f} = f_0 \cdot f_3^*$$
- Setting $a := \mathcal{F}(P_0)$, $b := \mathcal{F}(w \cdot h)$, $e := \exp(\alpha(\cdot)^2)$, this implies
$$f_0^* \cdot (a \cdot e + b) = f_0^* \cdot f = f_0 \cdot \tilde{f}^* = f_0 \cdot (a^* \cdot e^* + \tilde{b}^*)$$
- $e^* = \exp(\bar{\alpha}(\cdot)^2) \neq e \Rightarrow$ inconsistent lhs/rhs $\Rightarrow$ *Contradiction!* □

Conclusions

Physical Implications:

- ✓ Unique imaging of compact objects *from a single diffraction pattern!*
- ✓ Applicable to a large class of incident/background wave fields
- ✓ In general: Recovery of compactly perturbed paraxial wave fronts
- ✓ Relevant to QM by equivalence Schrödinger $\sim$ paraxial Helmholtz

Open Questions and Future Work:

- Phase retrieval may be severely ill-posed $\rightarrow$ *stability estimates*?
- Uniqueness robust under relaxation of approximations? Analogue in phaseless Helmholtz scattering? [**?**, **?**]
- Tailored regularization methods for numerical reconstructions

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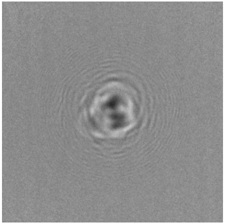
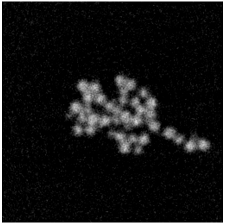
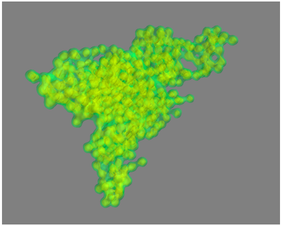
Proof of Concept: Experimental Data Set



Intensity data



δ: reconstructed slices



δ: 3D contour plot

- Colloidal Crystal of 415 nm polystyrene-beads
- Spherical shape and binary refractive index resolved
- $\beta \sim \frac{\delta}{2500} \sim 10^{-9}$ [Cloetens et al., 1999] $\rightarrow$ no absorption contrast!