

# Variational source conditions and stability estimates for inverse electromagnetic medium scattering problems

Frederic Weidling  
joint work with Thorsten Hohage

Institute for Numerical and Applied Mathematics  
Georg-August University of Göttingen

Inverse Problems: Modeling & Simulation  
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# Outline

- 1 Variational regularization theory
- 2 Problem description
- 3 Results
- 4 Proof via CGOs

# Tikhonov regularization

## General Setup

- Let  $\mathbb{X}, \mathbb{Y}$  be Banach spaces
- $F: \text{dom}(F) \subset \mathbb{X} \rightarrow \mathbb{Y}$  be a (possibly nonlinear) forward operator
- $f^\dagger \in \text{dom}(F)$  the true solution
- $g^\delta \in \mathbb{Y}$  observed data with  $\|g^\delta - F(f^\dagger)\|_{\mathbb{Y}} \leq \delta$

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**Tikhonov regularization:** Find an approximate solution

$$f_\alpha^\delta \in \arg \min_{f \in \text{dom}(F)} \left[ \frac{1}{\alpha} \|F(f) - g^\delta\|_{\mathbb{Y}}^2 + \Omega(f) \right],$$



where  $\Omega$  is an appropriate penalty term.

# Distance to true solution

Spectral source conditions:

$$f^\dagger = \varphi \left( F'[f^\dagger]^* F'[f^\dagger] \right) \omega$$

with an index function  $\varphi$ ,  $\rightsquigarrow$  additional restrictive requirements for nonlinear operators (tangential cone condition)



-  H. Engl, M. Hanke and A. Neubauer. *Regularization of inverse problems*, Kluwer, 1996.
-  P. Mathé and S. V. Pereverzev. *Geometry of linear ill-posed problems in variable Hilbert scales*. **Inverse Problems** 19:789–803, 2003.

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Variational source conditions (VSC):

$$\forall f \in \text{dom}(F) : \beta \Delta_\Omega(f, f^\dagger) \leq \Omega(f) - \Omega(f^\dagger) + \psi \left( \|F(f) - F(f^\dagger)\|_{\mathbb{Y}}^2 \right)$$

for a concave index function  $\psi$  and a  $\beta \in (0, 1]$ .

First used (with  $\psi(t) = c\sqrt{t}$ ) in

-  B. Hofmann, B. Kaltenbacher, C. Pöschl, and O. Scherzer. *A convergence rates result for Tikhonov regularization in Banach spaces with non-smooth operators*. **Inverse Problems** 23:987–1010, 2007.

# VSC vs. stability estimates

Let  $K \subset \text{dom}(F)$  be some smoothness class (e.g. a Sobolev ball), and  $\Delta$  a symmetric error measure.

Variational source condition:  $\forall f^\dagger \in K, f \in \text{dom}(F)$ :

$$\beta \Delta(f, f^\dagger) \leq \Omega(f) - \Omega(f^\dagger) + \psi \left( \|F(f) - F(f^\dagger)\|_{\mathbb{Y}}^2 \right)$$

Conditional stability estimate:  $\forall f_1, f_2 \in K$ :

$$\beta \Delta(f_1, f_2) \leq \psi \left( \|F(f_1) - F(f_2)\|_{\mathbb{Y}}^2 \right)$$

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Stability  $\stackrel{???}{\implies}$  VSC:

- $\Omega(f) - \Omega(f^\dagger)$  might be negativ
- VSC must hold on the larger set  $\text{dom}(F)$


# Advantages of VSCs


- simplify proofs, e.g. one can easily show that for optimal  $\alpha$  a VSC implies the convergence rate

$$\Delta_{\Omega}(f_{\alpha}^{\delta}, f^{\dagger}) \leq c\psi(\delta^2)$$

 M. Grasmair. *Generalized Bregman distances and convergence rates for non-convex regularization methods*. **Inverse Problems** 26:115014, 2010.

- for linear operators between Hilbert space even necessary conditions for certain rates of convergence

 J. Flemming, B. Hofmann, and P. Mathé. *Sharp converse results for the regularization error using distance functions*. **Inverse Problems**, 27:025006, 2011.

 T. Hohage, and F. Weidling. *Characterizations of variational source conditions, converse results, and maxisets of spectral regularization methods*. **arXiv** 1603.05133.

- no differentiability assumption  $\rightsquigarrow$  no restrictive assumption connecting  $F$  and  $F'$  needed (tangential cone condition)
- allow extension to Banach spaces and general data misfit/penalty terms

...but

But: few verifications so far

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Aim of this talk: **Verify a VSC** for inverse electromagnetic scattering under smoothness assumption

- Exploit similarities to stability estimates
- ↪ proceed similar to the case of acoustic scattering
  - 📖 T. Hohage and F. Weidling. *Verification of a variational source condition for acoustic inverse medium scattering problems*. **Inverse Problems**, 31:075006, 2015.

# Time-harmonic Maxwell equations

Time-harmonic Maxwell equations:

$$\nabla \times E - i\kappa H = 0$$

$$\nabla \times H + i\kappa n E = 0$$

with wave number  $\kappa$  and refractive index  $n$ :

$$\kappa := \omega \sqrt{\mu_0 \epsilon_0} \qquad n(x) := \frac{1}{\epsilon_0} \left( \epsilon(x) + i \frac{\sigma(x)}{\omega} \right)$$

- $\mu_0$  magnetic permeability, assumed to be constant
- $\epsilon(x) > 0$  electric permittivity, with  $\epsilon - \epsilon_0$  compactly supported
- $\sigma(x) \geq 0$  conductivity, compactly supported

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Direct Problem:

- Given incident field and refractive index  $n$  find radiating scattered field
- Solvable if  $n \in \mathfrak{D} := \{n \in C^{1,\alpha}(\mathbb{R}^3) : \text{supp}(1 - n) \subset B(\pi), \Re(n) > 0, \Im(n) \geq 0\}$

# Inverse near field problem

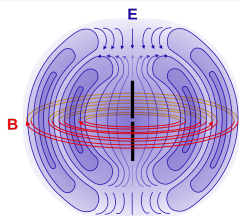


Image from wikipedia

Incident fields are generated by dipoles

$$E_{y,a}^i(x) = -\frac{1}{i\kappa} \nabla \times \nabla \times a\Phi(x,y)$$

$$H_{y,a}^i(x) = \nabla \times a\Phi(x,y)$$

for all  $y \in \partial B(R)$  with  $R > \pi$ ,  $a \in \mathbb{R}^3$  and  
 $\Phi(x,y) = e^{i\kappa|x-y|}/(4\pi|x-y|)$

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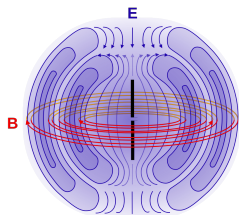


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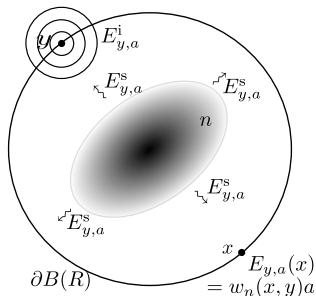
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Measurements are the matrices

$$E_{y,a}(x) = w_n(x,y)a$$

on  $\partial B(R) \times \partial B(R)$ .



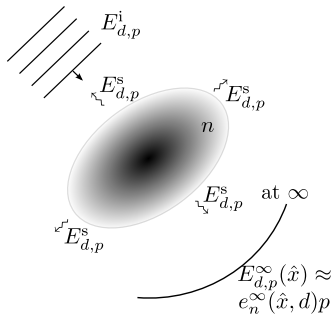
# Inverse far field problem

Incident fields are plane waves

$$E_{d,p}^i(x) = d \times (p \times d) e^{i\kappa d \cdot x}$$

$$H_{d,p}^i(x) = \frac{1}{i\kappa} \nabla \times E_{d,p}^i(x)$$

for all directions  $d \in \partial B(1)$  and polarization  $p \in \mathbb{R}^3$ .



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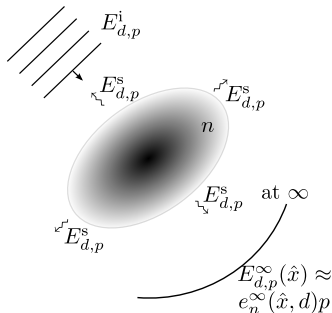
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Far field expansion

$$E_{d,p}^s(x) = \frac{e^{i\kappa r}}{r} (E_{d,p}^\infty(\hat{x}) + o(1))$$

as  $r = |x| \rightarrow \infty$  for measurement direction  $\hat{x} = x/r \in \partial B(1)$ .



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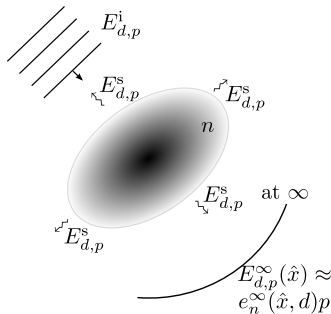
Far field expansion

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Measurements are the matrices defined by

$$E_{d,p}^\infty(\hat{x}) = e_n^\infty(\hat{x}, d)p$$



# Functional setup

Define

$$\mathfrak{D}_b := \{n \in C^{1,\alpha}(\mathbb{R}^3) : \text{supp}(1 - n) \subset B(\pi), \Re(n) \geq b, \Im(n) \geq 0\}$$

and let  $m > 7/2$  and set  $\mathbb{X} = H^m([-\pi, \pi]^3)$  with norm

$$\|f\|_{H^m}^2 = \sum_{\gamma \in \mathbb{Z}^3} \langle \gamma \rangle^m |\widehat{f}(\gamma)|^2 =: \Omega(f)$$

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Near field operator

$$F_n : H^m([-\pi, \pi]^3) \cap \mathfrak{D}_b \rightarrow (L^2(\partial B(R) \times \partial B(R)))^{3 \times 3}, \quad n \mapsto w_n.$$

Far field operator

$$F_f : H^m([-\pi, \pi]^3) \cap \mathfrak{D}_b \rightarrow (L^2(\partial B(1) \times \partial B(1)))^{3 \times 3}, \quad n \mapsto e_n^\infty.$$

# Main theorem


## Theorem

Assume that  $7/2 < m < s$ ,  $s \neq 2m + 3/2$  and  $n^\dagger \in \mathcal{D}_b$  satisfies  $\|n^\dagger\|_{H^s} \leq C_s$  for some  $C_s \geq 0$ . Then a VSC holds true for the operator  $F_n$  with  $\beta = 1/2$ , and  $\psi$  given by

$$\psi_n(t) := A (\ln(3 + t^{-1}))^{-2\nu}, \quad \nu := \min \left\{ \frac{s - m}{m + 5/2}, \frac{s - m}{s - m + 1} \right\},$$

where the constant  $A > 0$  depends on  $m, s, b, C_s, \kappa$  and  $R$ .

**Remark:** Instability results show optimality up to the value of the exponent

-  **N. Mandache.** *Exponential instability in an inverse problem for the Schrödinger equation.* **Inverse Problems**, 17:1435–1444, 2001.

# Corollaries

## Corollary (convergence rate)

*Under the assumptions of the previous theorem the error bound*

$$\|n_\alpha^\delta - n^\dagger\|_{H^m} \leq 2\sqrt{2A} (\ln(3 + \delta^{-2}))^{-\nu}$$

*holds true for Tikhonov regularization for optimal  $\alpha$ .*

## Corollary (stability estimate)

*Suppose  $\frac{7}{2} < m < s$ ,  $s \neq 2m + 3/2$  and  $n_1$  and  $n_2$  satisfy  $n_j \in \mathcal{D}_b$  with  $\|n_j\|_{H^s} \leq C_s$  for  $j = 1, 2$  and some  $C_s > 0$ . Then*

$$\|n_1 - n_2\|_{H^m} \leq \sqrt{2A} \left[ \ln \left( 3 + \|F_n(n_1) - F_n(n_2)\|_{(L^2(\partial B(R) \times \partial B(R)))^{3 \times 3}}^{-2}} \right) \right]^{-\nu}.$$

# VSC for far field data

## Theorem

*Under the assumptions of the previous theorem the operator  $F_f$  fulfills for all  $0 < \theta < 1$  a VSC with  $\beta = 1/2$  and  $\psi$  given by*




$$\psi_f(t) := B (\ln(3 + t^{-1}))^{-2\nu\theta}$$

*with a constant  $B > 0$  depending only on  $m, s, C_s, \kappa, b$  and  $\theta$ .*

**Remark:** One obtains similar corollaries as in the near field case.

# Stability result comparison

	<b>new</b>	<b>Hähler</b>	<b>Caro</b>	<b>Lai</b>
<b>data</b>	near and far field	far field	Cauchy	Cauchy
<b>validity</b>	global	local any-where	global	local around 0
<b>stability of</b>	$\sigma, \epsilon$	$\sigma, \epsilon$	$\sigma, \epsilon, \mu$	$\sigma$
<b>norm</b>	$H^m$	$L^\infty$	$H^1$	$H^{-s}$
<b>min smooth</b>	$H^s$ , $s > m > 7/2$	$C^{2,\gamma}$ , $\gamma > 0$	$H^{2+\gamma} \cap W^{2,\infty}$ , $\gamma > 0$	$H^{2s+2}$ $s > 3/2$
<b>exponent</b>	$< 1$	1/15	unknown, $< 1/3$	$\leq 1$
<b>special</b>		strong norm image space		Hölder-logarithmic

-  P. Hähler. *Stability of the inverse electromagnetic inhomogeneous medium problem.* **Inverse Problems**, 16:155–174, 2000.
-  P. Caro. *Stable determination of the electromagnetic coefficients by boundary measurements.* **Inverse Problems**, 26:105014, 2010.
-  R.-Y. Lai, V. Isakov and J.-N. Wang. *Increasing stability for the conductivity and attenuation coefficient.* arXiv:1505.00108, 2015.

# Proof in a nutshell

- Reformulate VSC to

$$\langle n^\dagger, n^\dagger - n \rangle_{H^m} \leq \frac{1}{4} \|n^\dagger - n\|_{H^m}^2 + \psi(\delta)$$

- Show VSC holds independently of  $\delta$  outside of a ball
- Split in high and low frequencies
  - Use higher smoothness of true solution for high frequencies
  - Use Alessandrini type estimate and CGOs to bound low frequencies
- Choose occurring parameter in dependence of  $\delta$  such that right hand side is approximately minimal

# Alessandrini type estimate


Connection between **potentials** and **data**

## Lemma

Let  $2R > R' > R > \pi$ ,  $m > 7/2$  and assume that  $n_1$  and  $n_2$  are two refractive indices satisfying  $n_j \in \mathcal{D}_b \cap H^m$  such that  $\|n_j\|_{H^m} \leq C_m$  for some  $C_m \geq 0$ . Let  $E_j, H_j \in C^1(B(2R)) \cap L^2(B(2R))$  be solutions to the pdes in  $B(2R)$  for  $n = n_j$  for  $j = 1, 2$ . Then the estimate

$$\left| \int_{B(\pi)} (n_1 - n_2) E_1 E_2 \, dx \right| \\ \leq C \|w_1 - w_2\|_{L^2} \|E_1\|_{L^2(B(R'))} \|E_2\|_{L^2(B(R'))}$$

holds true, where  $w_j$  is the near field scattering data for  $n = n_j$  for  $j = 1, 2$  and  $C$  depends on  $\kappa, R, b$  and  $C_m$ .

-  P. Hähner. *Stability of the inverse electromagnetic inhomogeneous medium problem*. **Inverse Problems**, 16:155–174, 2000.

# General idea

Desired solutions  $E_j$  are of the form

$$E_j = \eta_j e^{i\zeta_j \cdot x} \quad \zeta_j \in \mathbb{C}^3 \quad \zeta_1 = \overline{\zeta_2} \quad \zeta_j \cdot \zeta_j = \kappa^2$$

to obtain bounds on Fourier coefficient  $\zeta_1 + \zeta_2$ .

## Inhomogeneous case

- Not possible
- Aim: **construct solutions close** to these
- Strategy: **transformation to Helmholtz** like equation


# Helmholtz type equation

Construct **Helmholtz type** equation for  $(E', H') = (n^{1/2}E, H)$  with  $E^i = \eta e^{i\zeta \cdot x}$

$$(\Delta + \kappa^2) \begin{pmatrix} E' \\ H' \end{pmatrix} = \mathcal{Q} \begin{pmatrix} E' \\ H' \end{pmatrix}$$

$$\begin{aligned} \mathcal{Q}(x) := & \kappa^2(1 - n)\mathbf{1}_6 + i\kappa n^{-1/2} \begin{pmatrix} 0_3 & -\nabla n \times \\ \nabla n \times & 0_3 \end{pmatrix} - \begin{pmatrix} D\left(\frac{\nabla n}{n}\right) & 0_3 \\ 0_3 & 0_3 \end{pmatrix} \\ & - \frac{1}{4}n^{-2} \begin{pmatrix} \mathbf{1}_3(\nabla n \cdot \nabla n) & 0_3 \\ 0_3 & 0_3 \end{pmatrix} + \frac{1}{2}n^{-1} \begin{pmatrix} \mathbf{1}_3(\Delta n) & 0_3 \\ 0_3 & 0_3 \end{pmatrix} \end{aligned}$$

In order to apply the usual CGO theory one needs to bound  $\|\mathcal{Q}\|_2$  for  $n \in \mathfrak{D}_b \cap H^m$ ,  $\|n\|_{H^m} \leq C_m$ .

 **D. Colton and L. Päivärinta.** *The uniqueness of a solution to an inverse scattering problem for electromagnetic waves.* **Archive for rational mechanics and analysis**, 119:59–70, 1992.

# Form and estimate of CGOs

For  $\Im(\zeta) \geq C \|Q\|_2$  one obtains **existence** of solution to the time-harmonic Maxwell equations of the form

$$E(x, \zeta, \eta) = e^{i\zeta \cdot x} [\eta + f(x, \zeta, \eta)\zeta + V(x, \zeta, \eta)], \quad x \in B(2R)$$

fulfilling the **estimate**

$$\|f(\cdot, \zeta, \eta)\|_{L^2(B(R'))} + \|V(\cdot, \zeta, \eta)\|_{L^2(B(R'))} \leq C \frac{|\eta|}{|\Im(\zeta)|}.$$

# Inserting in Alessandrini estimate

## Lemma

Let  $R > \pi$ ,  $m > 7/2$  and  $n_1$  and  $n_2$  be two refractive indices such that  $n_j \in \mathfrak{D}_b \cap H^m$  such that  $\|n_j\|_{H^m} \leq C_m$  with  $C_m \geq 0$  and corresponding near field data  $w_j$  for  $j = 1, 2$ . Let

$$t \geq C(1 + \kappa^2)C_m^2 b^{-2}$$

and  $1 \leq \varrho \leq 2\sqrt{\kappa^2 + t^2}$ . Then there exists a constant  $C$  depending only on  $R, \kappa, b$  and  $C_m$  such that

$$|(\hat{n}_1 - \hat{n}_2)(\gamma)| \leq C \left( \|w_1 - w_2\|_{L^2} e^{3Rt} + \|n_1 - n_2\|_{H^m} \frac{\varrho}{t} \right)$$

holds true for all  $\gamma \in \mathbb{Z}^3$  with  $|\gamma| \leq \varrho$ .

**Remark:** The better estimate on  $\|\mathcal{Q}\|_2$  improves previous results which had an additional factor of  $\varrho^3$ .

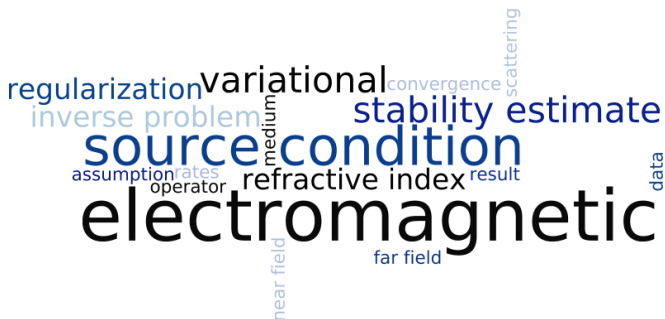
# Summary


- First proof of (logarithmic) convergence rates for Tikhonov applied to electromagnetic medium scattering under Sobolev smoothness
- A competitive stability estimate for electromagnetic medium scattering
- Proof
  - improves parameter dependence of previous results
  - shows that ideas of the proof of the VSC for acoustic medium scattering can be applied to other problems

but ...

- We cannot guarantee that a global minimum of the Tikhonov functional can be computed. This still requires  $F'$  and conditions such as the tangential cone condition.

# Thank you for your attention.



 F. Weidling and T. Hohage. *Variational source conditions and stability estimates for inverse electromagnetic medium scattering problems*, [arxiv](https://arxiv.org/abs/1512.06586), 1512.06586, 2015.