

Verification of a Variational Source Condition for Inverse Medium Scattering

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Outline

- 1 Variational regularization theory
- 2 Results on inverse medium scattering
- 3 Proof ideas
- 4 Conclusion

Tikhonov regularization

General Setup

- Let \mathbb{X}, \mathbb{Y} be Banach spaces
- $F: \text{dom}(F) \subset \mathbb{X} \rightarrow \mathbb{Y}$ be a (possibly nonlinear) forward operator
- $f^\dagger \in \text{dom}(F)$ the true solution
- $g^\delta \in \mathbb{Y}$ observed data with $\|g^\delta - F(f^\dagger)\|_{\mathbb{Y}} \leq \delta$

Tikhonov regularization

$$f_\alpha^\delta \in \arg \min_{f \in \text{dom}(F)} \left[\frac{1}{\alpha} \|F(f) - g^\delta\|_{\mathbb{Y}}^2 + \frac{1}{2} \|f\|_{\mathbb{X}}^2 \right]$$

Bound $\|f_\alpha^\delta - f^\dagger\|_{\mathbb{X}}$

Spectral source conditions:

$$f^\dagger = \varphi (F'[f^\dagger]^* F'[f^\dagger]) \omega$$

with an index function φ , i.e. $\varphi: [0, \infty) \rightarrow [0, \infty)$ is non-decreasing and $\varphi(0) = 0$.


 H. Engl, M. Hanke and A. Neubauer. *Regularization of inverse problems*, Kluwer, 1996.

Variational source conditions (VSC):

$$\forall f \in \text{dom}(F) : \frac{\beta}{2} \|f^\dagger - f\|_{\mathbb{X}}^2 \leq \frac{1}{2} \|f\|_{\mathbb{X}}^2 - \frac{1}{2} \|f^\dagger\|_{\mathbb{X}}^2 + \psi \left(\|F(f) - F(f^\dagger)\|_{\mathbb{Y}}^2 \right)$$

for a concave index function ψ and a $\beta \in (0, 1]$.

First used (with $\psi(t) = c\sqrt{t}$) in


 B. Hofmann, B. Kaltenbacher, C. Pöschl, and O. Scherzer. *A convergence rates result for Tikhonov regularization in Banach spaces with non-smooth operators*. **Inverse Problems** 23:987–1010, 2007.

Advantages of VCSs


- simplify proofs, e.g. one can easily show that a VSC implies the convergence rate

$$\beta \|f_\alpha^\delta - f^\dagger\|_{\mathbb{X}}^2 \leq 4\psi(\delta^2)$$

for the optimal choice of the regularization parameter α

 M. Grasmair. *Generalized Bregman distances and convergence rates for non-convex regularization methods*. **Inverse Problems** 26:115014, 2010.

- for linear operators between Hilbert space even necessary conditions for certain rates of convergence

 J. Flemming, B. Hofmann, and P. Mathé. *Sharp converse results for the regularization error using distance functions*. **Inverse Problems**, 27:025006, 2011.

- no differentiability assumption \rightsquigarrow no restrictive assumption connecting F and F' needed (tangential cone condition)
- allow extension to Banach spaces and general data misfit/penalty terms

...but

Verification of VSCs:

- Reformulations of VSC with $\psi(x) = \sqrt{x}$ for a phase retrieval and an option pricing problem.
 - 📖 B. Hofmann, B. Kaltenbacher, C. Pöschl, and O. Scherzer. *A convergence rates result for Tikhonov regularization in Banach spaces with non-smooth operators*. **Inverse Problems** 23:987–1010, 2007.
- Spectral source conditions imply VSCs.
- For linear operators F with ℓ^q penalty term via the range of F^*
 - 📖 M. Burger, J. Flemming, and B. Hofmann. *Convergence rates in ℓ^1 -regularization if the sparsity assumption fails*. **Inverse Problems**, 29:025013, 2013.
 - 📖 S. W. Anzengruber, B. Hofmann, and R. Ramlau. *On the interplay of basis smoothness and specific range conditions occurring in sparsity regularization*. **Inverse Problems**, 29:125002, 2013.
 - 📖 J. Flemming and M. Hegland. *Convergence rates in ℓ^1 -regularization when the basis is not smooth enough*. **Applicable Analysis**, 94:464–476, 2015.

↪ few verifications so far

VSC vs. stability estimates

Let $K \subset \text{dom}(F)$ be some smoothness class (e.g. a Sobolev ball).

Conditional stability estimate: $\forall f_1, f_2 \in K$

$$\beta \|f_1 - f_2\|_{\mathbb{X}}^2 \leq \psi \left(\|F(f_1) - F(f_2)\|_{\mathbb{Y}}^2 \right)$$

Variational source condition: $\forall f^\dagger \in K, f \in \text{dom}(F)$:

$$\beta \|f^\dagger - f\|_{\mathbb{X}}^2 \leq \|f\|_{\mathbb{X}}^2 - \|f^\dagger\|_{\mathbb{X}}^2 + \psi \left(\|F(f) - F(f^\dagger)\|_{\mathbb{Y}}^2 \right)$$

VSC \implies Stability:

- W.l.o.g. $\|f_1\|_{\mathbb{X}} \geq \|f_2\|_{\mathbb{X}}$, choose $f_1 = f^\dagger$, $f_2 = f$

Stability $\stackrel{???}{\implies}$ VSC:

- sign of $\|f\|_{\mathbb{X}}^2 - \|f^\dagger\|_{\mathbb{X}}^2$ unknown
- VSC must hold on the larger set $\text{dom}(F)$

Forward medium scattering

Given a **refractive index** n and one/several initial wave(s) u^i fulfilling the Helmholtz equation $\Delta u^i + \kappa^2 u^i = 0$ we want to find the total field(s) $u = u^i + u^s$ solving

$$\begin{aligned} \Delta u + \kappa^2 n u &= 0 && \text{in } \mathbb{R}^3, \\ \frac{\partial u^s}{\partial r} - i\kappa u^s &= \mathcal{O}(r^{-2}) && \text{as } r = |x| \rightarrow \infty. \end{aligned}$$

for a fixed wave number κ .

Assumption on the contrast $f = 1 - n$

$$f \in \mathcal{D} := \{f \in L^\infty(\mathbb{R}^3) : \text{supp}(f) \subset B(\pi), \Re(f) \leq 1, \Im(f) \leq 0\}$$

with $B(R) := \{x \in \mathbb{R}^3 : |x| \leq R\}$.

Near field inverse problem

Incident fields are point source waves

$$u_y^i(x) = \frac{1}{4\pi} \frac{e^{i\kappa|x-y|}}{|x-y|}$$

for all $y \in \partial B(R)$ with $R > \pi$.

Measurements are the total fields

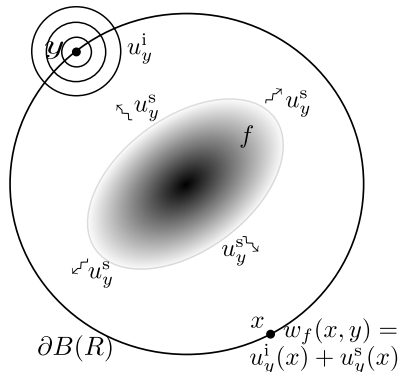
$$w_f(x, y) = u_y^i(x) + u_y^s(x)$$

on $\partial B(R) \times \partial B(R)$.

Functional Setting:

$$F_n: \mathfrak{D} \cap H_0^m(B(\pi)) \rightarrow L^2(\partial B(R) \times \partial B(R)), \quad f \mapsto w_f$$

We choose $\mathfrak{X} = H_0^m(B(\pi))$ the Sobolev space of order m for $m > 3/2$.



Main theorem


Theorem

Assume that $3/2 < m < s$, $s \neq 2m + 3/2$ and $f^\dagger \in \mathfrak{D}$ satisfies $\|f^\dagger\|_{H^s} \leq C_s$ for some $C_s \geq 0$. Then a VSC holds true for the operator F_n with $\beta = 1/2$, and ψ given by

$$\psi_n(t) := A (\ln(3 + t^{-1}))^{-2\mu}, \quad \mu := \min \left\{ 1, \frac{s - m}{m + 3/2} \right\},$$

where the constant $A > 0$ depends only on m, s, C_s, κ and R .

Remark: Theorem holds true for any $\beta \in (0, 1)$, but ψ_n depends on β .

 T. Hohage, F. Weidling. *Verification of a variational source condition for acoustic inverse medium scattering problems*. **Inverse Problems**. 31:075006, 2015.

Corollaries

Corollary (convergence rate)

Under the assumptions of the previous theorem the error bound

$$\|f_\alpha^\delta - f^\dagger\|_{H^m} \leq 2\sqrt{2A} (\ln(3 + \delta^{-2}))^{-\mu}$$

holds true for Tikhonov regularization for optimal α

Corollary (stability estimate)

Suppose $\frac{3}{2} < m < s$, $s \neq 2m + 3/2$ and f_1 and f_2 satisfy $f_j \in \mathcal{D}$ with $\|f_j\|_{H^s} \leq C_s$ for $j = 1, 2$ and some $C_s > 0$. Then

$$\|f_1 - f_2\|_{H^m} \leq \sqrt{2A} \left[\ln \left(3 + \|F_n(f_1) - F_n(f_2)\|_{L^2(\partial B(R)^2)}^{-2} \right) \right]^{-\mu}.$$

Far field inverse problem

Incident fields are plane waves

$$u_{d,\infty}^i(x) = e^{i\kappa d \cdot x}$$

for all $d \in \partial B(1)$.

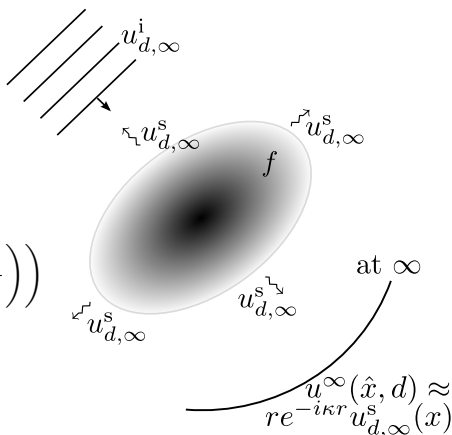
Measurements is the far field pattern defined by

$$u(x) = u^i(x) + \frac{e^{i\kappa r}}{r} \left(u^\infty(\hat{x}) + \mathcal{O}\left(\frac{1}{r^2}\right) \right)$$

as $r = |x| \rightarrow \infty$ measured in all directions $\hat{x} = x/r \in B(1)$.

Functional Setting:

$$F_f: \mathcal{D} \cap H_0^m(B(\pi)) \rightarrow L^2(\partial B(1) \times \partial B(1)), \quad f \mapsto u^\infty$$



VSC for far field data


Theorem

Under the assumptions of the previous theorem the operator F_f fulfills for all $0 < \theta < 1$ a VSC with $\beta = 1/2$ and ψ given by

$$\psi_f(t) := B (\ln(3 + t^{-1}))^{-2\mu\theta}$$

with a constant $B > 0$ depending only on m, s, C_s, κ and θ .


Remark: One obtains similar corollaries as in the near field case.

 T. Hohage, F. Weidling. *Verification of a variational source condition for acoustic inverse medium scattering problems*. **Inverse Problems**. 31:075006, 2015.

Known results (... very incomplete list)


Instability results

show that logarithmic estimates are essentially optimal


-  N. Mandache. *Exponential instability in an inverse problem for the Schrödinger equation*. **Inverse Problems**, 17:1435–1444, 2001.

Stability results


local stability, strong norm in image space, unknown coefficient $\mu \leq 1$:

-  P. Stefanov. *Stability of the inverse problem in potential scattering at fixed energy*. **Ann. Inst. Fourier (Grenoble)**, 40:867–884, 1990.

global stability, $\mathbb{Y} = L^2$ and explicit $\mu \leq 1$:

-  P. Hähner and T. Hohage. *New stability estimates for the inverse acoustic inhomogeneous medium problem and applications*. **SIAM J. Math. Anal.**, 33:670–685, 2001.

explicit dependence of κ yielding asymptotically Hölder stability and $\mu \rightarrow \infty$ as $s \rightarrow \infty$

-  M. I. Isaev and R. G. Novikov. *Effectivized Hölder-logarithmic stability estimates for the Gel'fand inverse problem*. **Inverse Problems**, 30:095006, 2014.

Reformulation

Choice of norm on $H_0^m(B(\pi))$:

$$\|f\|_{H^m}^2 := \sum_{\gamma \in \mathbb{Z}^3} (1 + |\gamma|^2)^m \left| \widehat{f}(\gamma) \right|^2$$

where $\widehat{f}(\gamma) = (2\pi)^{-3/2} \int_{C(\pi)} f(x) e^{-i\gamma \cdot x} dx$ are the Fourier coefficients of f in the cube $C(\pi) = (-\pi, \pi)^3$ with f extended by 0 outside of $B(\pi)$.

Equivalent formulation of the variational source condition:

$$\begin{aligned} \Re \sum_{\gamma \in \mathbb{Z}^3} (1 + |\gamma|^2)^m \widehat{f^\dagger}(\gamma) \overline{\left(\widehat{f^\dagger}(\gamma) - \widehat{f}(\gamma) \right)} &= \Re \langle f^\dagger, f^\dagger - f \rangle_{H^m} \\ &\leq \frac{1 - \beta}{2} \|f^\dagger - f\|_{H^m}^2 + \psi \left(\|F_n(f) - F_n(f^\dagger)\|_{\mathbb{Y}}^2 \right) \end{aligned}$$

The simple part...

A universal bound: Assume that $\|f^\dagger - f\|_{H^m} \geq 4C_s$, then

$$\begin{aligned} \Re \langle f^\dagger, f^\dagger - f \rangle_{H^m} &\leq \|f^\dagger\|_{H^m} \|f^\dagger - f\|_{H^m} \\ &\leq C_s \|f^\dagger - f\|_{H^m} \leq \frac{1}{4} \|f^\dagger - f\|_{H^m}^2 \end{aligned}$$

\rightsquigarrow VSC independent of δ

Bound on high frequencies: Let $\varrho > 0$, then

$$\Re \sum_{\gamma \in \mathbb{Z}^3 \setminus B(\varrho)} (1 + |\gamma|^2)^m \widehat{f^\dagger}(\gamma) \overline{(\widehat{f^\dagger}(\gamma) - \widehat{f}(\gamma))} \leq \frac{1}{8} \|f^\dagger - f\|_{H^m}^2 + 2C_s \varrho^{2(m-s)}$$


by Cauchy-Schwarz's and Young's inequality.

Low frequencies

Lemma

Let $\pi < R < R'$. Then there exists a positive constant C such that for all contrasts $f_1, f_2 \in \mathcal{D}$ with corresponding near fields w_1 and w_2 , and for all solutions $u_j \in H^2(B(R'))$ to $\Delta u_j + \kappa^2 u_j = \kappa^2 f_j u_j$ the following estimate holds true:

$$\left| \int_{B(\pi)} (f_1 - f_2) u_1 u_2 dx \right| \leq C \|w_1 - w_2\|_{L^2(\partial B(R)^2)} \|u_1\|_{L^2(B(R'))} \|u_2\|_{L^2(B(R'))}$$

 P. Hähner and T. Hohage. New stability estimates for the inverse acoustic inhomogeneous medium problem and applications. **SIAM J. Math. Anal.**, 33:670–685, 2001.


Note: If we could choose $u_j = \exp(i\zeta_j \cdot x)$ with $\zeta_j \in \mathbb{C}^3$, $\zeta_1 = \overline{\zeta_2}$ and $\zeta_j \cdot \zeta_j = \sum_{l=1}^3 \zeta_{j,l}^2 = \kappa^2$ we would get bounds on Fourier coefficients of $f_1 - f_2$.

CGO solutions

Solutions to $\Delta u + \kappa^2 u = \kappa^2 f u$ of the form

$$u(x) = e^{i\zeta \cdot x} (1 + v(x)) \quad \text{with } \zeta \in \mathbb{C}^3, \zeta_1 \cdot \zeta_2 = \sum_{j=1}^3 \zeta_j^2 = \kappa^2, t := |\Im(\zeta)| > 0$$

with "small" v are called **Complex Geometric Optics solutions**. For existence and estimates

 **P. Hähner**. A Periodic Faddeev-Type Solution Operator. *Journal of Differential Equations*, 128:300–308, 1996.

Lemma

Let $C_m > 0$, $m > 3/2$, $\pi < R < R'$ and f_1 and f_2 be contrasts with $f_j \in \mathcal{D}$ and $\|f_j\|_{H^m} \leq C_m$ with corresponding near field data w_j for $j = 1, 2$. Let $t \geq t_0(\kappa, C_m, \dots)$ and $1 \leq \varrho \leq 2\sqrt{\kappa^2 + t^2}$. Then there exists a constant $C > 0$ such that for all $\gamma \in \mathbb{Z}^3 \cap B(\varrho)$ we have

$$\left| \widehat{f}_1(\gamma) - \widehat{f}_2(\gamma) \right| \leq C e^{4R't} \|w_1 - w_2\|_{L^2(\partial B(R)^2)} + \frac{C}{t} \|f_1 - f_2\|_{H^m}.$$

Conclusion

- First rigorous verification of a variational source condition for a nonlinear inverse problem in PDEs
- First rigorous proof of (logarithmic) convergence rates for Tikhonov applied to medium scattering under Sobolev smoothness
- Proof uses techniques from stability estimates

but ...

- We cannot guarantee that a global minimum of the Tikhonov functional can be computed. This still requires F' and conditions such as the tangential cone condition.

Thank you for your attention

